Affine Schubert Calculus

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Introductory Workshop:
Combinatorial Algebraic Geometry

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$\tilde{\mathrm{Fl}}$: affine flag ind-variety $\tilde{\mathrm{Gr}}$: affine Grassmannian   (type A)

ring	Schubert basis
$H^*(\tilde{\mathrm{Fl}})$	affine Schubert polynomial
$H^*(\tilde{\mathrm{Gr}})$	affine Schur (dual k-Schur)
$H_*(\tilde{\mathrm{Gr}})$	k-Schur

- 1997: Dale Peterson's MIT lectures; quantum equals affine Grassmannian homology Schubert calculus
- 2003: Lapointe, Lascoux, Morse: k-Schur functions
- 2004: Conjecture: k-Schurs are the homology Schubert basis of affine Grassmannian.
- 2005: Lapointe, Morse: dual k-Schur functions
- 2005: Thomas Lam: affine Stanley functions
- 2006: Lam proves conjecture
- 2015: Seung Jin Lee: affine Schubert polynomials

Affine flags and Affine Grassmannian

$G = GL_n(\mathbb{C}) \supset B \supset T$	upper triangular, diagonal
$\mathrm{Fl}_n \cong G/B$	Flag variety
$\mathbb{C}[[z]] = \{a_0 + a_1 z + a_2 z^2 + \cdots \mid a_i \in \mathbb{C}\}$	Formal power series
$\mathbb{C}((z)) = \bigcup_{m \geq 0} z^{-m}\mathbb{C}[[z]] = \mathrm{Frac}(\mathbb{C}[[z]])$	Formal Laurent series
$\tilde{G} = GL_n(\mathbb{C}((z)))$	Affine Kac-Moody group
$\tilde{P} = GL_n(\mathbb{C}[[z]])$	Maximal Parahoric subgroup
$\tilde{B} = \{f(z) \in \tilde{P} \mid f(0) \in B\}$	Iwahori subgroup
$\tilde{\mathrm{Fl}} = \tilde{G}/\tilde{B}$	affine flag ind-variety
$\tilde{\mathrm{Gr}} = \tilde{G}/\tilde{P}$	affine Grassmannian
$\mathrm{Fl}_n \cong \tilde{P}/\tilde{B}$	
$\tilde{\mathrm{Fl}}^T \cong \tilde{S}_n$	affine Weyl group

$\tilde{S}_n$ as Coxeter group

Generators: s_i for $i \in \tilde{I} = \{0, 1, 2, \ldots, n-1\} = \mathbb{Z}/n\mathbb{Z}$

Relations:

$$\begin{aligned} s_i^2 &= 1 & & \\ s_i s_j s_i &= s_j s_i s_j & & \text{if } n \geq 3 \text{ and } j \in \{i \pm 1\} \\ s_i s_j &= s_j s_i & & \text{if } j \notin \{i \pm 1\} \end{aligned}$$

Reduced word for $w \in \tilde{S}_n$: $a_1 a_2 \cdots a_\ell$ with $a_j \in \tilde{I}$ such that

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell}$$

with ℓ minimum $\qquad \ell(w) = \ell$

$\tilde{S}_n \supset S_n =$ subgroup generated by s_i for $i \in I = \{1, 2, \ldots, n-1\}$

$\tilde{S}_n$ as semidirect product

S_n acts on coroot lattice
$Q^\vee = \{(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n \mid \beta_1 + \beta_2 + \cdots + \beta_n = 0\}$

$$Q^\vee \hookrightarrow \tilde{S}_n$$
$$\beta \mapsto t_\beta$$

$$\tilde{S}_n \cong Q^\vee \rtimes S_n$$
$$u t_\beta u^{-1} = t_{u(\beta)} \qquad u \in S_n,\ \beta \in Q^\vee$$

$$s_0 \mapsto t_{\theta^\vee} s_\theta$$
$$\theta^\vee = (1, 0, \ldots, 0, -1) \in Q^\vee$$
$$s_\theta = (1, n) \qquad \text{swaps 1 and } n$$
$$= s_1 s_2 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1 \qquad \textit{reduced}$$

Lifting $\tilde{S}_n$ to $\tilde{G}$

$$\begin{aligned}
\tilde{S}_n &\hookrightarrow \tilde{G} = GL_n(\mathbb{C}((z))) \\
u &\mapsto \text{permutation matrix of } u && \text{for } u \in S_n \\
t_\beta &\mapsto \operatorname{diag}(z^{\beta_1}, z^{\beta_2}, \ldots, z^{\beta_n}) && \text{for } \beta \in Q^\vee
\end{aligned}$$

Cell decomposition of $\tilde{\mathrm{Fl}}$ and cohomology basis

$w \in \tilde{S}_n \subset \tilde{G}$

$\tilde{\mathrm{Fl}} \supset X_w^\circ := \tilde{B} w \tilde{B} / \tilde{B} \cong \mathbb{C}^{\ell(w)}$	Schubert cell
$X_w = \overline{X_w^\circ}$	Schubert variety
$\tilde{\mathrm{Fl}} = \bigsqcup_{w \in \tilde{S}_n} X_w^\circ$	cell decomp.
$H^*(\tilde{\mathrm{Fl}}) \cong \bigoplus_{w \in \tilde{S}_n} \mathbb{Q}[X_w]$	Schubert basis

Recall

$$\tilde{\mathrm{Fl}} = \tilde{G}/\tilde{B}$$
$$\tilde{\mathrm{Gr}} = \tilde{G}/\tilde{P}$$
$$\mathrm{Fl}_n \cong \tilde{P}/\tilde{B}$$

There is a ring isomorphism

$$H^*(\tilde{\mathrm{Fl}}) \cong H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\mathrm{Fl}_n)$$

Reason:
$\tilde{\mathrm{Fl}}$ is equal to $\tilde{\mathrm{Gr}} \times \mathrm{Fl}_n$ “in the limit”

Find explicit presentation of $H^*(\tilde{\mathrm{Fl}})$ and Schubert basis.

Borel's presentation of $H^*(\mathrm{Fl}_n)$ and Lascoux and Schützenberger's Schubert polynomials $\mathfrak{S}_w$

$$\begin{aligned}
\mathbb{Q}[X_n] &= \mathbb{Q}[x_1, x_2, \ldots, x_n] \\
\mathbb{Q}[X_n]^{S_n} &= \text{symmetric polynomials} \\
J &= (f \in \mathbb{Q}[X_n]^{S_n} \mid f(0) = 0) \qquad \text{ideal} \\
H^*(\mathrm{Fl}_n) &\cong \mathbb{Q}[X_n]/J \\
&= \bigoplus_{w \in S_n} \mathbb{Q}\mathfrak{S}_w
\end{aligned}$$

Symmetric functions

Hopf algebra Λ of symmetric functions in $x_+ = (x_1, x_2, x_3, \dots)$

$$\Lambda = \mathbb{Q}[p_1, p_2, p_3, \dots]$$

$$p_k(x_+) = \sum_{i \geq 1} x_i^k$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \qquad \text{for partition } \lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{Y}$$

$$\Lambda = \bigoplus_{\lambda \in \mathbb{Y}} \mathbb{Q} p_\lambda$$

$$\langle p_\lambda \, , \, p_\mu \rangle = z_\lambda \delta_{\lambda\mu} \qquad \text{Hall pairing}$$

$$z_\lambda = \prod_i i^{m_i} m_i! \qquad \lambda \text{ has } m_i \text{ parts } i$$

$$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k \qquad \text{for all } k \geq 1.$$

Dual Hopf algebras $H_*(\tilde{\mathrm{Gr}})$ and $H^*(\tilde{\mathrm{Gr}})$

$$\Lambda_n = \mathbb{Q}[p_1, p_2, \ldots, p_{n-1}] = \mathbb{Q}[h_1, h_2, \ldots, h_{n-1}] \subset \Lambda$$
$$\Lambda^n = \Lambda/(p_n, p_{n+1}, p_{n+2}, \ldots) = \Lambda/(m_\lambda \mid \lambda_1 \geq n)$$

$$\begin{array}{ccc} \Lambda \otimes \Lambda & \xrightarrow{\langle\cdot\,,\cdot\rangle} & \mathbb{Q} \\ \uparrow \quad \downarrow & & \uparrow \\ \Lambda_n \otimes \Lambda^n & \xrightarrow[\langle\cdot\,,\cdot\rangle]{} & \mathbb{Q} \end{array}$$

Λ_n and Λ^n are dual

Both have $\mathbb{Q}$-basis $\{p_\lambda \mid \lambda \in \mathbb{Y},\ \lambda_1 \leq n-1\}$

There are Hopf isomorphisms [Bott]

$$H_*(\tilde{\mathrm{Gr}}) \cong \Lambda_n$$
$$H^*(\tilde{\mathrm{Gr}}) \cong \Lambda^n$$

Presentation of $H^*(\tilde{\mathrm{Fl}})$

There is a ring isomorphism

$$\begin{aligned} H^*(\tilde{\mathrm{Fl}}) &\cong H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\mathrm{Fl}_n) \\ &\cong \Lambda^n \otimes \mathbb{Q}[X_n]/J. \end{aligned}$$

Goal: Characterize Schubert basis in $\Lambda^n \otimes \mathbb{Q}[X_n]/J$.

Action of $\tilde{S}_n$ on $\Lambda^n \otimes \mathbb{Q}[X_n]$

Use $\Lambda(x_-)$: symmetric functions in $x_- = (x_0, x_{-1}, x_{-2}, \dots)$

$\Lambda^n(x_-) \otimes \mathbb{Q}[X_n]$ is a polynomial $\mathbb{Q}$-algebra with generators

$$p_1(x_-), p_2(x_-), \dots, p_{n-1}(x_-)$$
$$x_1, x_2, \dots, x_n$$

S_n permutes $\mathbb{Q}[X_n]$ and leaves $\Lambda(x_-)$ invariant.

For $\beta = (\beta_1, \dots, \beta_n) \in Q^\vee$, t_β leaves $\mathbb{Q}[X_n]$ invariant and

$$t_\beta(p_k) = p_k + \sum_{j=1}^{n} \beta_j x_j^k.$$

This well-defines an action of $\tilde{S}_n$ on $\Lambda^n(x_-) \otimes \mathbb{Q}[X_n]$ and on $\Lambda^n(x_-) \otimes \mathbb{Q}[X_n]/J$.

Since $s_0 = t_{\theta^\vee} s_\theta$ with $\theta^\vee = (1, 0, \ldots, 0, -1)$ and $s_\theta = (1, n)$

$$\begin{aligned} s_0(p_k) &= t_{\theta^\vee} s_\theta(p_k) = t_{\theta^\vee}(p_k) \\ &= p_k + x_1^k - x_n^k \\ s_0(x_1) &= t_{\theta^\vee} s_\theta(x_1) = t_{\theta^\vee}(x_n) = x_n \\ s_0(x_n) &= x_1 \\ s_0(x_j) &= x_j \qquad \text{for } 2 \leq j \leq n-1. \end{aligned}$$

Divided difference operators A_i on $\Lambda^n \otimes \mathbb{Q}[X_n]$

$$\alpha_i = x_i - x_{i+1} \qquad \text{for } 1 \leq i \leq n-1$$

$$\alpha_0 = x_n - x_1$$

$$A_i(f) = \alpha_i^{-1}(f - s_i(f))$$

Proposition

1. *The image of A_i and the kernel of A_i are the s_i-invariant elements. $A_i^2 = 0$ for all $i \in \tilde{I}$.*
2. $A_iA_j = A_jA_i$ *for all* $j \notin \{i \pm 1\}$.
3. $A_iA_jA_i = A_jA_iA_j$ *if* $n \geq 3$ *and* $j \in \{i \pm 1\}$.
4. $A_i(fg) = A_i(f)g + s_i(f)A_i(g)$.
   If $s_i(f) = f$ *then* $A_i(fg) = fA_i(g)$.
5. *The* A_i *are well-defined on* $\Lambda^n \otimes \mathbb{Q}[X_n]/J$.

Affine Schubert polynomials [Seung Jin Lee 2015]

Theorem

There is a unique basis $\{\tilde{\mathfrak{S}}_w \mid w \in \tilde{S}_n\}$ of $\Lambda^n(x_-) \otimes \mathbb{Q}[x]/J$ such that

1. $\tilde{\mathfrak{S}}_{\mathrm{id}} = 1$.
2. $\tilde{\mathfrak{S}}_w$ *is homogeneous of degree* $\ell(w)$ *for all* $w \in \tilde{S}_n$.
3. *For all* $w \in \tilde{S}_n$ *and* $i \in \tilde{I}$

$$A_i(\tilde{\mathfrak{S}}_w) = \begin{cases} \tilde{\mathfrak{S}}_{ws_i} & \text{if } \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover

$$\begin{aligned} \Lambda^n(x_-) \otimes \mathbb{Q}[X_n]/J &\cong H^*(\tilde{\mathrm{Fl}}) \\ \tilde{\mathfrak{S}}_w &\mapsto [X_w]. \end{aligned}$$

Schubert polynomials for $H^*(\tilde{\mathrm{Gr}})$: affine Schur/dual k-Schur functions

Say that $w \in \tilde{S}_n$ is **affine Grassmannian** (denoted $w \in \tilde{S}_n^0$) if

$$\ell(ws_i) > \ell(w) \qquad \text{for all } i \in I = \{1, 2, \ldots, n-1\}.$$

Equivalently, either $w = \mathrm{id}$ or every reduced word for w ends in 0.

Min length representatives of cosets $\tilde{S}_n / S_n$.

$$H^*(\tilde{\mathrm{Gr}}) \cong H^*(\tilde{\mathrm{Fl}})^{S_n} \cong \bigoplus_{w \in \tilde{S}_n^0} \mathbb{Q}[X_w].$$

The S_n-invariant subring of $\Lambda(x_-) \otimes \mathbb{Q}[X_n]/I$ is $\Lambda(x_-)$.

Corollary

$\{\tilde{\mathfrak{S}}_w \mid w \in \tilde{S}_n^0\}$ *is a basis of* $\Lambda^n(x_-)$.

Schubert polynomials for $H_*(\tilde{\mathrm{Gr}})$: k-Schur functions

For $w \in \tilde{S}_n^0$ let

$$\{\tilde{\mathfrak{S}}_w^\vee \in \Lambda_n \mid w \in \tilde{S}_n^0\}$$

be the basis of Λ_n dual to the basis

$$\{\tilde{\mathfrak{S}}_w \mid w \in \tilde{S}_n^0\}$$

of Λ^n under $\langle \cdot\,,\,\cdot \rangle : \Lambda_n \otimes \Lambda^n \to \mathbb{Q}$.

For $u, v, w \in \tilde{S}_n^0$ define the affine Grassmannian Schubert homology structure constants $d_{uv}^w \in \mathbb{Q}$ by

$$\tilde{\mathfrak{S}}_u^\vee \tilde{\mathfrak{S}}_v^\vee = \sum_{w \in \tilde{S}_n^0} d_{uv}^w \tilde{\mathfrak{S}}_w^\vee.$$

Theorem (Peterson 1997)

Quantum Equals Affine Theorem. The d_{uv}^w are the Schubert structure constants of the quantum cohomology of Fl_n.