Quantum cohomology of homogeneous spaces II

Bootcamp ICERM – 03.02.2021.

Goals of this second lecture:

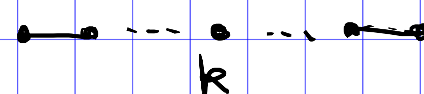
* give some techniques for computing GW-inv^{ts}
* Define QK quantum K-theory.

Computing degree 1 invariants: counting lines

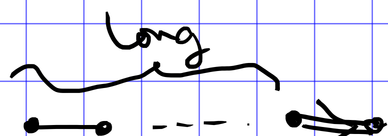
Assumption: $X = G/P$ with $P \subset G$ maximal parabolic

ie $\operatorname{Pic} X = \mathbb{Z}$ $\Leftrightarrow$ classes of curves given by the degree.

$\boxed{\text{AND}}$ P corresponds to a long simple root.

Examples: All $Gr(k,n)$: 

all roots are long.

B_n : 

eg $OG(k, 2n+1)$

C_n : 

$IG(k, 2n)$ is not OK.

Assumption $\Rightarrow$ lines are easy to describe.

<u>Thm</u> [Landsberg - Manivel, Strickland]

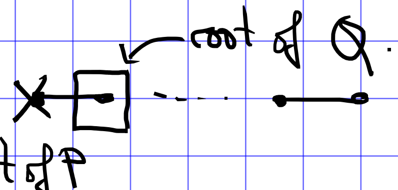
Let $Y_1 = F(X)$ = Fano variety of lines = {lines in X}

then Y_1 is <u>G-homogeneous</u> and projective so

$$Y_1 = G/Q \text{ for some } Q \subset G \text{ parabolic}$$

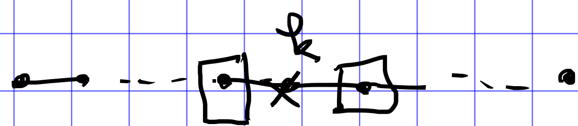
The simple roots ass. to Q are obtained as the roots adjacent to the root of P in the Dynkin diagram.

<u>Ex</u>: (1) $X = \mathbb{P}^{n-1}$ so type A_{n-1} :



so $Y_1 = F(X) = G/Q = Gr(2, n)$ = lines in $\mathbb{P}^{n-1}$ (OK!)

(2) $X = Gr(k, n)$



$$Y_1 = F(X) = G/Q = Fl(k-1, k+1; n).$$

Geometry: this says that $\overline{M}_{0,1}(X,1)$ and $\overline{M}_{0,0}(X,1)$ are easy to describe:

$$\begin{array}{ccc} \overline{M}_{0,1}(X,1) \xrightarrow{ev} X = G/P & & Z_1 = G/_{P\cap Q} \longrightarrow G/P \\ \downarrow & = & \downarrow \\ \overline{M}_{0,0}(X,1) = F(X) & & Y_1 = G/Q \end{array}$$

Ex: $Fl(k-1,k,k+1;n) \longrightarrow X = Gr(k,n)$

$$\downarrow$$

$Fl(k-1,k+1;n)$

But recall: GW-invariants are enumerative.

$$\langle \sigma^u, \sigma^v, \sigma^w \rangle_{X,1} = \#\{\text{lines meeting } g_1 X^u, g_2 X^v, g_3 X^w\}$$

for $g_1, g_2, g_3 \in G$ general.

Fact: $\{\text{lines meeting } X^u\} = q_1 \underbrace{p_1^{-1}(X^u)}_{\{(x,\ell)\ \mid\ \ell \ni x \in X^u\}}$

$$\begin{array}{ccc} \{pt \subset \ell\} = Z_1 & \xrightarrow{p_1} & X \\ \downarrow & & \\ Y_1 = \{\text{line } \ell\} & & \end{array}$$

But all this is G-equivariant so $q_1 p_1^{-1}(X^u)$ is B-stable and irred $\Rightarrow$ it is a Schubert variety.

$\Rightarrow$ $\{\text{lines meeting } X^u\} = q_1 p_1^{-1}(X^u) = Y^{\hat{u}}$ for some $\hat{u}$.

Corollary:

$$\langle \sigma^u, \sigma^v, \sigma^w \rangle_{X,1} = \#\{\text{lines meeting } g_1 X^u, g_2 X^v, g_3 X^w\}$$
$$= \#\ g_1 Y^{\hat{u}} \cap g_2 Y^{\hat{v}} \cap g_3 Y^{\hat{w}}$$
$$= \deg\left([Y^{\hat{u}}] \cup [Y^{\hat{v}}] \cup [Y^{\hat{w}}]\right)$$

$\Rightarrow$ quantum-to-classical principle: GW-invariants of deg 1 are classical (deg 0) invariants on another homo. space.

$\rightsquigarrow$ generalisations in higher degree (with some constants!) works well for $Gr(k,n)$.

Quantum K-theory:

First: K-theory it is a generalisation of cohomology (and a special case of oriented cohomology $\leadsto$ cobordism)

We assume X is smooth $/\mathbb{C}$.

$$K(X) = \frac{\text{free } \mathbb{Z}\text{-module gen by } [\mathcal{F}] \ \mathcal{F} \text{ loc free sheaf}}{([\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''] \text{ for } 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0)}$$

$$\overset{X \text{ smooth}}{=} \frac{\text{free } \mathbb{Z}\text{-module gen by } [\mathcal{F}] \ \mathcal{F} \text{ coh. sheaf}}{(\text{same rel}^{\circ})}$$

Also $=$ Groth gp of $D^b(X)$.

Ex: For $X = G/P$ we have $(X_u)_{u \in W^P}$ Schubert var

$$X_u \leadsto \mathcal{O}_{X_u} \leadsto \mathcal{O}_u = [\mathcal{O}_{X_u}] \in K(X)$$

$$K(X) = \bigoplus_{u \in W^P} \mathbb{Z}\mathcal{O}_u \quad (\text{compare with cohomology}).$$

for $f : X \to Y$ we have $f^* : K(Y) \to K(X)$ pull-back

for $f : X \to Y$ proper we have $f_* : K(X) \to K(Y)$.

Furthermore $K(X)$ is a ~~ring~~ ring via

$$[\mathcal{F}]\cdot[\mathcal{G}] = [\mathcal{F}\otimes\mathcal{G}]$$ on loc. free sheaves.

Ex: For Y, Z in X st Y and Z intersect transversely

then $[\mathcal{O}_Y]\cdot[\mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}]$

Ex: $Y = \mathbb{P}^2 \subset \mathbb{P}^3$ plane $Z = Q_2 \subset \mathbb{P}^3$ quadric

then $[\mathcal{O}_Y]\cdot[\mathcal{O}_Z] = [\mathcal{O}_{\text{conic}}] = [\mathcal{O}_{X\text{ 2 lines}}]$

But $[\mathcal{O}_{\overline{X\text{ 2 lines}}}] = [\mathcal{O}_{\text{line}}] + [\mathcal{O}_{\text{line}}] - [\mathcal{O}_{pt}]$

because of the exact sequence:

$$0 \to \mathcal{O}_{X\text{ lines}} \to \mathcal{O}_{\text{line}} \oplus \mathcal{O}_{\text{line}} \to \mathcal{O}_{pt} \to 0.$$

Quantum K-theory:

$$\langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{O}_w \rangle_{X,d} = \pi_*\left(ev_1^* \mathcal{O}_u \cdot ev_2^* \mathcal{O}_v \cdot ev_3^* \mathcal{O}_w\right) \in K(pt) = \mathbb{Z}.$$

also

$$\langle \mathcal{O}_u, \mathcal{O}_v \rangle_{X,d} = \pi_*\left(ev_1^* \mathcal{O}_u \cdot ev_2^* \mathcal{O}_v\right).$$

Recall for QH:

$$\sigma_u * \sigma_v = \sum_{d,w} q^d \langle \sigma_u, \sigma_v, (\sigma_w)^\vee \rangle \sigma_w$$

← Poincaré dual.

if we set:

$$\mathcal{O}_u * \mathcal{O}_v = \sum_{d,w} q^d \langle \mathcal{O}_u, \mathcal{O}_v, (\mathcal{O}_w)^\vee \rangle \mathcal{O}_w.$$

this is not associative.

You need to replace this by:

Def:

$$\mathcal{O}_u * \mathcal{O}_v = \sum_{d,w} q^d N_{uv}^{wd} \mathcal{O}_w.$$

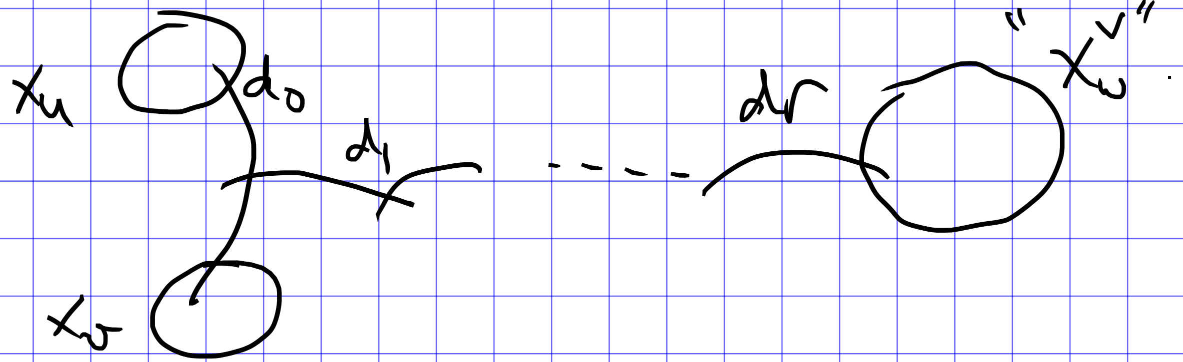
with:

$$N^{w,d}_{u,v} = \sum_{r} \sum_{\substack{d_0+\dots+d_r=d \\ k_1,\dots,k_r}} (-1)^r \langle O_u, O_v, O_{k_1}\rangle_{d_0} \left(\prod_{i=1}^{r-1} \langle O_{k_i}, O_{k_{i+1}}\rangle_{d_i}\right) \langle O_{k_r}, O^{\vee}_{w}\rangle_{d_r}$$

$$\left(N^{w,d}_{u,v} \right)_{u,v,w,d} \Longleftrightarrow \left(\langle O_u, O_v, O_w \rangle_{X,d} \right)_{u,v,w,d}.$$

This is to ~~be~~ take into account the geometry of $\overline{M}_{0,3}(X,d)$ and especially curves that degenerate and become reducible

the above sum "counts" curves of the form



The sum with signs is because the boundary in $\overline{\mathcal{M}}_{0,3}(X,d)$ is a SND: