Quantum cohomology of homogeneous spaces I

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- Quantum cohomology has many aspects. In these lectures, I will insist on 2 of them:

  * deformation of $H^*(X,\mathbb{C})$
  * enumerative geometry.

These aspects are of course related and are also related to many others that will not show-up in these lectures (eg. Integrable systems).

1) Reminders on homogeneous spaces:

$X = G/P$ with G reductive and $P \subset G$ a parabolic subgroup (ie G/P is G-homogeneous & projective)

These varieties are very well understood. For ex. we have:

* $\mathrm{Pic}(X) = \mathbb{Z}^r = \bigoplus_{i=1}^{r} \mathbb{Z} D_i$ — D_i generate the ample cone (giving all embeddings)
* $A_1(X) = \text{curves} = \mathbb{Z}^r = \bigoplus_{i=1}^{r} \mathbb{Z} C_i$ — C_i generate effective curves (true curves).

These space are dual for intersection: $\langle\ ,\ \rangle$ the pairing and $\langle D_i , C_j \rangle = \delta_{ij}$. (Poincaré dual basis).

• <u>Cohomology:</u> X has a cellular decomposition, the <u>Bruhat decomposition</u> given by B-orbits for $B \subset G$ a Borel subgp (eg, for $G = GL_n$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$)

$$B^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

We have: $X = G/P = \coprod_{w \in W^P} BwP/P = \coprod_{w \in W^P} B^- wP/P$

W^P some indexing set (connected to the Weyl gr of G).

Schubert cells: $\Omega_w = BwP/P$ opposite $\Omega^w = B^- wP/P$

$\ell(w) = \dim \Omega_w = \operatorname{codim} \Omega^w$.

Schubert varieties $X_w = \overline{\Omega_w}$ and $X^w = \overline{\Omega^w}$

Schubert cells $\sigma_w = [X_w]$ and $\sigma^w = [X^w]$.

These classes σ_w, σ^w live in $H^*(X, \mathbb{Q})$ form basis

$$(\sigma_w)_w \longleftrightarrow (\sigma^w)_w$$ that are dual for Poincaré duality.

<u>Ex:</u> For $\ell(w) = 1$ we get:

$\sigma_w = [C_i]$ $i = 1, \dots, r$ classes of curves.
Schubert curves.

$\sigma^w = [D_i]$ $i = 1, \dots, r$ Schubert divisors.

2) Rational curves.

Quantum cohomology counts rational curves. I will use a simplified version that is enough in this special case:

Fix $d \in A_1(X)$ a class of curves, we have

$$d = \sum_{i=1}^{r} d_i C_i \quad \text{with } d_i \geq 0 \quad \left(\begin{array}{l}\text{so that there are true}\\ \text{curves of class } d\end{array}\right)$$

Set $M_{0,3}(X,d) = \{ f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = d \}$.

Note: (1) we have up to automorphisms 3 marked pts: $0, 1, \infty$.

(2) eval$^\circ$ maps: $ev_{1,2,3} : M_{0,3}(X,d) \to X$, $f \mapsto f(0), f(1), f(\infty)$.

(3) these maps are G-equivariant.

Quantum cohomology = intersection theory on $M_{0,3}(X,d)$.

Pb: not compact.

Solution: Choose any smooth compactification $\overline{M}_{0,3}(X,d)$ such that $ev_{1,2,3}$ extend and stay G-equiv.

(this is possible by blowing-up).

Definition (Gromov-Witten invariants).

For $\alpha, \beta, \gamma \in H^*(X, \mathbb{C})$ define:

$$\langle \alpha, \beta, \gamma \rangle_{X,d} = \pi_* \left(ev_1^* \alpha \cup ev_2^* \beta \cup ev_3^* \gamma \right)$$

where $\pi : \overline{M}_{0,3}(X, d) \longrightarrow pt = \mathrm{Spec}(\mathbb{C})$.

Remark: By the properties of π_* we have

$$\langle \alpha, \beta, \gamma \rangle_{X,d} = 0 \text{ unless}$$

$$\deg \alpha + \deg \beta + \deg \gamma = \dim \overline{M}_{0,3}(X, d) = \langle -K_X, d \rangle + \dim X.$$

(here K_X = canonical divisor on X).

Looks bad? did not get the def?

Not a pb, these invariants are enumerative.

Thm (Kleiman-Bertini) If $e : M \to Z$ is a map with Z homogeneous (here $M = \overline{M}_{0,3}(X,d)$, $e = ev$, $Z = X^3$, $\Gamma = G^3$) for a gp Γ and if $Y \subset Z$ has codim c, is reduced / smooth, then $e^{-1}(gY)$ is empty or has codim c is reduced / smooth.

Corollary: for u, v, w st $\deg\sigma^u + \deg\sigma^v + \deg\sigma^w = \dim \overline{M}_{0,3}(X, d)$

then
$$\langle \sigma^u, \sigma^v, \sigma^w \rangle_{X,d} = \#\{ f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = d \text{ and } f(0) \in g_1 X^u, f(1) \in g_2 X^v, f(\infty) \in g_3 X^w \}$$

for $g_1, g_2, g_3 \in G^3$ general.

Remark: with this we can compute a lot of GW-invariants (eg all of $\mathbb{P}^n$, see exercises!).

Remark: For $d = 0$, we have curves of class 0 so pts.

So $\langle \sigma^u, \sigma^v, \sigma^w \rangle_{X,0} = \deg(\sigma^u \cup \sigma^v \cup \sigma^w)$ (classical int° invariant).

3) Quantum cohomology:

These invariants patch together to form an algebra that is a deformation of $(H^*(X,\mathbb{C}), \cup)$.

Recall that $A_1(X) = \bigoplus_{i=1}^{r} \mathbb{Z} C_i$. For each i, let q_i be a variable and for $d = \sum d_i C_i$, set

$$q^d = q_1^{d_1} \cdots q_r^{d_r}.$$

Def: As $\mathbb{C}$-vectorspace $QH(X) = H^*(X,\mathbb{C})[q_1, \dots, q_r]$

$$= H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[q_1, \dots, q_r].$$

Define a product of 2 elts σ^u, σ^v by:

Poincaré duals

$$\sigma^u * \sigma^v = \sum_{d \in A_1(X)} \sum_{w} q^d \langle \sigma^u, \sigma^v, \sigma_w \rangle_{X,d} \, \sigma^w.$$

Remark:
$$(\sigma^u * \sigma^v)_{q=0} = \sum_{\substack{w \\ \text{w } d=0}} \langle \sigma^u, \sigma^v, \sigma_w \rangle_{X,0} \, \sigma^w$$
$$= \sum_{w} (\sigma^u \cup \sigma^v \cup \sigma_w) \, \sigma^w$$
$$= \sigma^u \cup \sigma^v.$$

So $*$ deforms $\cup$.

Thm [Kontsevich] $*$ is associative (and commutative).
The algebra is graded with $\deg q_i = \langle -K_X, C_i \rangle$.

(some) Pbms on $QH(X)$

- give a presentation
- give products with generators (Pieri)
- give classes in terms of generators (Giambelli)
- compute products ...

4) Presentation

Knowing a presentation of $H^*(X,\mathbb{Q})$ helps (a lot!).

Thm [Siebert–Tian] Assume we have a presentation

$$H^*(X,\mathbb{Q}) = \mathbb{Q}[\sigma_1, \dots, \sigma_s] \big/ (f_1 \cdots f_t)$$

with σ_i and f_j homogeneous.

Then $QH^*(X) = \mathbb{C}[\sigma_1 \cdots \sigma_s, q_1 \cdots q_r] \big/ (\tilde{f}_1 \cdots \tilde{f}_t)$

with $\deg \tilde{f}_j = \deg f_j$ and

$$\tilde{f}_j \big|_{q=0} = f_j .$$

<u>Example</u>: (1) $X = \mathbb{P}^n$, $H^*(X, \mathbb{C}) = \mathbb{C}[h] \big/ (h^{n+1})$

h = hyperplane class

$$QH(X) = \mathbb{Q}[h, q] \big/ (h^{n+1} - q) \qquad \deg q = n+1$$

(2) $X = Gr(k, n)$

$$H^*(X, \mathbb{C}) = \mathbb{C}[x_1 \cdots x_n]^{S_k \times S_{n-k}} \big/ (e_1 \cdots e_n)$$

$e_i = i^{th}$ sym. function in $x_1 \cdots x_n$.

Then $QH(X) = \mathbb{C}[x_1, \dots, x_n, q]^{S_k \times S_{n-k}} \Big/ (e_1, \dots, e_n - \lambda q)$

$\deg q = n$.

$\lambda \in \mathbb{C}$ (actually $\lambda = \pm 1$)

(3) $X = GL_n/B$ complete flags.

$$H^*(X, \mathbb{C}) = \mathbb{C}[x_1 \dots x_n] \Big/ (e_1 \dots e_n).$$

$$QH(X) = \mathbb{C}[x_1 \dots x_n, q_1 \dots q_{n-1}] \Big/ (\tilde{e}_1, \dots, \tilde{e}_n)$$

where

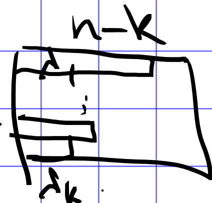
$$A = \begin{pmatrix} x_1 & 1 & & \\ q_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & q_{n-1} & x_n \end{pmatrix}$$

$\tilde{e}_i$ = coef of char poly of A

5) Pieri formula for $Gr(k,n)$.

$X = Gr(k,n)$.

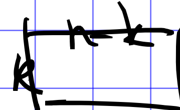
Def: A partition is a sequence $\lambda_1 \geq \dots \geq \lambda_r \geq 0$
$\lambda = (\lambda_1, \dots, \lambda_r, \dots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$.

- write $\lambda \subset k$ 

if $\lambda_r = 0$ for $r > k$ and $\lambda_1 \leq n-k$.

Recall: $(\sigma^w)_{w \in W^P}$ basis of $H^*(X, \mathbb{C})$

$$W^P \overset{1:1}{\longleftrightarrow} \{\lambda \text{ partition } \lambda \subset k\ \text{[box]}\}$$



We write σ^λ the corresp. Schubert class.

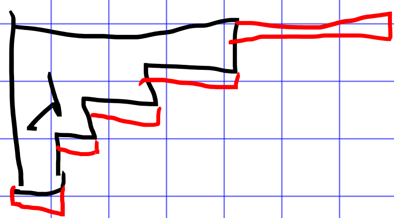
Special classes: $\lambda = (p) = (p, 0 \dots 0)$.

Prop: the classes $(\sigma^{(p)})_{1 \leq p \leq k}$ are generators of $H^*(X, \mathbb{C})$.
(Giambelli).

$$\sigma^\lambda = \det(\sigma^{\lambda_i + i - j})_{1 \leq i,j \leq k}.$$

Product with the $\sigma^{(p)}$:

Set $\Pi_\lambda^p = \{$ partitions μ obtained from λ by adding p boxes with no 2 on the same column $\}$



$\square$ = where you can add boxes.

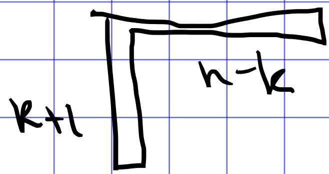
Thm (Pieri)

$$\sigma^{(p)} \cup \sigma^\lambda = \sum_{\substack{\mu \subset \boxed{k \times (n-k)} \\ \mu \in \Pi_\lambda^p}} \sigma^\mu .$$

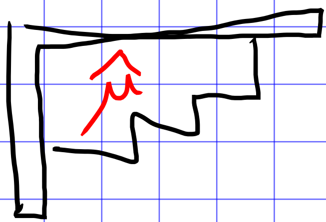
Thm (Quantum-Pieri).

$$\sigma^{(p)} * \sigma^\lambda = \sum_{\substack{\mu \subset \boxed{k \times (n-k)} \\ \mu \subset \Pi_\lambda^p}} \sigma^\mu + q \sum_{\substack{\mu \in \Pi_\lambda^p \\ \mu \in \boxed{(k+1) \times (n-k)}}} \sigma^{\hat{\mu}}$$

Here $\hat{\mu}$ exists **only if** $\mu \subset$ 

contains a full **hook** 

and

$\mu =$  $\hat{\mu} = \mu - \text{hook}$.

Thm: Giambelli formula stays unchanged.

Thm: $QH(X) = \mathbb{Q}[\sigma^{(1)}, \dots, \sigma^{(k)}, q] \Big/ (Y_{k+1}, \dots, Y_{n-1}, Y_n - \lambda q)$

$\lambda = \pm 1$

$Y_\ell = \det(\sigma_{\lambda_i + i - j})_{1 \le i,j \le \ell} \qquad \ell \in [k+1, n]$.