Introduction to Chow rings of matroids

Chris Eur (TA: Lukas Kühne, Connor Simpson, Mariel Supina)

§1. Matroids

Defn A *matroid* $M=(\overset{\text{ground set}}{E},\overset{\text{bases}}{\mathcal{B}})$ of rank r on $E=\{0,1,\dots,n\}$ is a collection of r-subsets $\mathcal{B}\subseteq\binom{E}{r}$ such that:

$$\forall B_1,B_2\in\mathcal{B} \text{ and } i\in B_1\setminus B_2,\ \exists j\in B_2\setminus B_1 \text{ st } B_1\setminus\{i\}\cup\{j\}\in\mathcal{B}.$$

E.g. ① G graph with edges E ⇝ $M(G)=(E,\mathcal{B})$ with
$\mathcal{B}=$ spanning forests
= maximal acyclic subsets of edges

☆ Linear matroid ② $\{f_0,\dots,f_n\}$ spanning k-vect. sp. $L^\vee$ of dim $=r$ ⇝ $M(L)=(E,\mathcal{B})$ with $\{i_1,\dots,i_r\}\in\mathcal{B}$ iff $\{f_{i_1},\dots,f_{i_r}\}$ basis of $L^\vee$.

↕

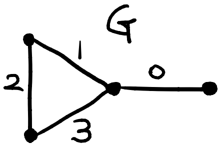
$k^{n+1}\twoheadrightarrow L^\vee$

↕

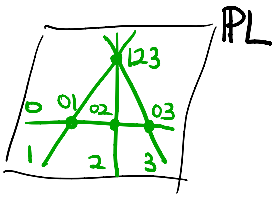
$L\hookrightarrow k^{n+1}$ ⇝ hyperplane arr. $\mathbb{P}L_i=\{v\in\mathbb{P}L\mid f_i(v)=0\}$

$(\mathbb{P}L\hookrightarrow\mathbb{P}^n)$ $=\mathbb{P}L\cap$ (ith coord. hyperplane of $\mathbb{P}^n$)

E.g.

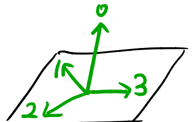


$\{\mathbb{P}L_i\}_i\subset\mathbb{P}L\subset\mathbb{P}^3$



$\mathbb{P}L_F=\{v\in\mathbb{P}V\mid f_i(v)=0\ \forall i\in F\}.$

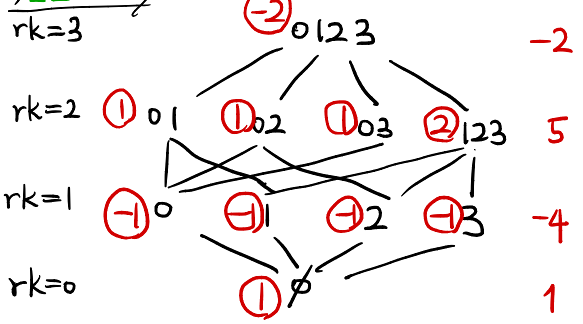
$L^\vee\simeq k^3$



$$\begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&-1\end{bmatrix}$$

$M(G)=M(L)$

$\mathcal{B}=\{012,\ 013,\ 023\}.$



$$\chi_M(q) = q^3-4q^2+5q-2 = (q-1)^2(q-2)$$

Defn $\mathrm{rk}_M: 2^E\longrightarrow\mathbb{Z}$ *rank function* by $\mathrm{rk}_M(I)=\max\limits_{B\in\mathcal{B}}|I\cap B|.$

$\mathrm{rk}_{M(L)}(I)=\dim_k \operatorname{span}(f_i\mid i\in I).$

$F\subseteq E$ is a *flat* if $\mathrm{rk}_M(F\cup e)>\mathrm{rk}_M(F)$ $\forall e\in E\setminus F.$

$\chi_M(q)$ *characteristic polynomial*

$\approx \chi_G(q)$ chromatic polynomial $=\#(q\text{-proper colorings})$ when $M=M(G).$

§2. Wonderful compactification

wond. cpt.

(r-1)-dim'l

$$\begin{tikzcd} W_L := \widetilde{\mathbb{P}L} \arrow[r, hook] \arrow[d, "\pi"'] & X_{A_n} \arrow[d] \arrow[dr] & \\ \mathbb{P}L \arrow[r, hook] & \mathbb{P}^n \arrow[r, "crem"] & \mathbb{P}^n \end{tikzcd}$$

X_{A_n} = permutohedral variety

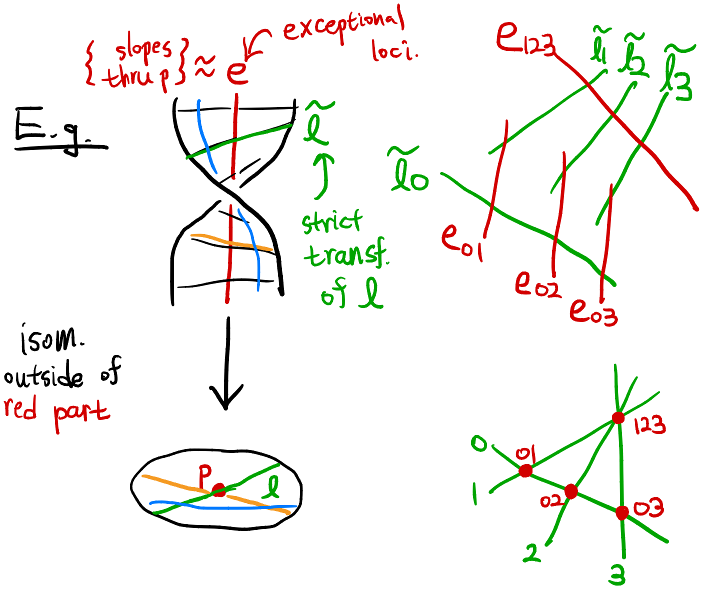
$[x_0 : \cdots : x_n] \mapsto [\frac{1}{x_0} : \cdots : \frac{1}{x_n}]$.

$W_L = \mathbb{P}L$ blown-up at $\mathbb{P}L_F$ $\mathrm{rk}_M(F) = r-1$ (pts)

then blown-up at (strict transf. of) $\mathbb{P}L_F$ $\mathrm{rk}(F) = r-2$

$\vdots$

E.g.



$$\begin{tikzcd} W_L \arrow[d, "\pi"'] \\ \mathbb{P}L \supset [h] \end{tikzcd}$$

$\alpha = [\pi^{-1}(h)]$

$x_0 = [\tilde{l}_0]$ $x_{03} = [e_{03}]$

$\mathbb{P}L$ is $\mathbb{P}^2$ here.

$[h]$ any hyperplane (line here).

x_0x_1, $x_{01}x_{02}$, x_0x_{123}, ⋯

$\alpha = x_1 + x_{01} + x_{123}$

$= x_0 + x_{01} + x_{02} + x_{03}$

$\vdots$

$$\deg(\alpha\beta) = \deg(\alpha(x_0 + x_1 + x_2 + x_3 + x_{01} + x_{02} + x_{03} + x_{123} - \alpha))$$

$$1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 - 1 = 3$$

Chow ring of M

Defn $A^\bullet(M) = \dfrac{\mathbb{R}[x_F \mid \phi \subsetneq F \subsetneq E \text{ flat}]}{\langle x_F x_{F'} \mid F, F' \text{ incomp} \rangle + \langle \sum_{i \in F} x_F - \sum_{j \in F'} x_{F'} \mid i, j \in E \rangle}$.

if $M = M(L)$ $\|$ $A^\bullet(W_L)$

Thm Let $\alpha = \sum_{i \in F} x_F$, $\beta = \sum_{i \notin F'} x_{F'} = (\sum_F x_F) - \alpha \in A^\bullet(M)$.

(1) $\deg : A^{r-1}(M) \xrightarrow{\sim} \mathbb{R}$ via $\deg(\alpha^{r-1}) = 1 = \deg(\underbrace{x_{F_1} \cdots x_{F_{r-1}}}_{\text{max'l flag of flats}})$ $(\deg = \deg_{W_L})$

(2) We have $(\deg(\alpha^{r-1}), \deg(\alpha^{r-2}\beta), \cdots, \deg(\beta^{r-1}))$

= unsigned coeff. of $\chi_M(q)/(q-1)$.

Exer 0. Verify $\beta^2 = 2$, noting that

$$\chi_M(q) = (q-1)(q-2) = q^2 - 3q + 2.$$

References

Surveys: Baker: Hodge theory in combinatorics
Huh: Tropical geometry of matroids
Katz: Matroid theory for algebraic geometers

General: Fulton: Introduction to toric varieties (Ch. 5)
Oxley: Matroid Theory
Lazarsfeld: Positivity in AG (Ch. 1)
Maclagan-Sturmfels: Introduction to tropical geometry (Ch. 6)

Articles: Adiprasito-Huh-Katz ‘18
De Concini-Procesi ‘95)
Feichtner-Yuzvisnky ‘04) wnd. cpt.
Backman-Eur-Simpson ‘19) Exer 3.
Proudfoot-Speyer ‘06) Exer 2.(c).
… many many more…