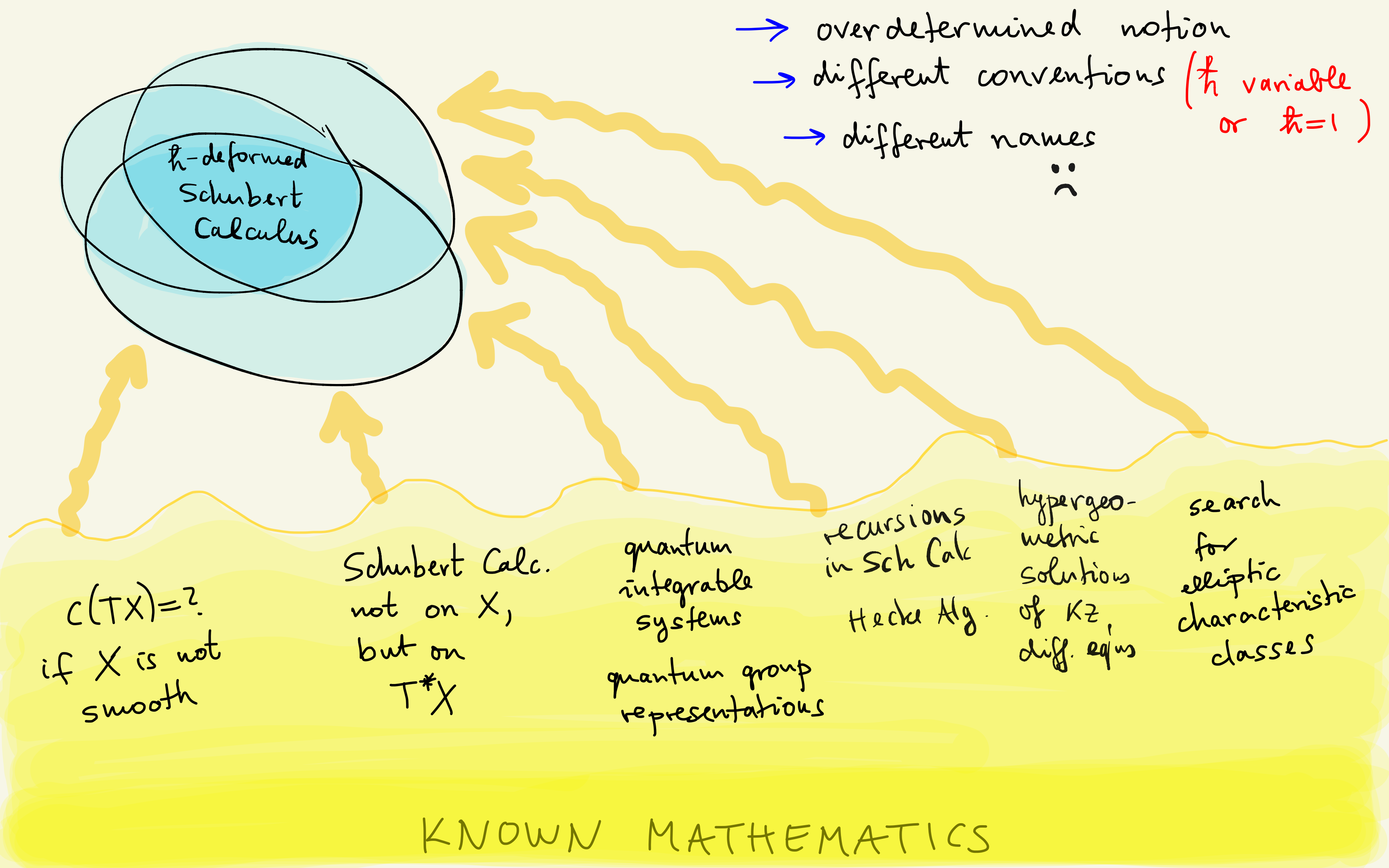
Introduction

+ $H^*_T(Gr)$



→ overdetermined notion
→ different conventions ($\hbar$ variable or $\hbar=1$)
→ different names
:(
$\hbar$-deformed Schubert Calculus
$c(TX)=?$ if X is not smooth
Schubert Calc. not on X, but on T^*X
quantum integrable systems
quantum group representations
recursions in Sch Calc
Hecke Alg.
hypergeometric solutions of KZ, diff. eq'ns
search for elliptic characteristic classes
KNOWN MATHEMATICS

GEOMETRY

$$T \circlearrowright \left(\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathbb{C}^n \arrow[r] \arrow[d] & Q \arrow[dl] \\ & Gr_k\mathbb{C}^n & \end{tikzcd} \right)$$ bundles

$Gr_k\mathbb{C}^n \supset \Omega_I$

Schubert cells

ALGEBRA-COMBIN.

$H_T^*(Gr_k\mathbb{C}^n)$

$\ni [\overline{\Omega}_I]$ fundamental class of closure of Ω_I
(aka as Schubert class)

$\ni c^{sm}(\Omega_I)$ $\hbar$-deformed Sch class
(aka Chern-Schwartz-MacPherson class)

Ex $k=1$ $n=2$

$H^*_T(Gr_1 \mathbb{C}^2) =$

$$\left\{ (f_1(z_1, z_2), f_2(z_1, z_2)) \in \mathbb{Z}[z_1, z_2] \times \mathbb{Z}[z_1, z_2] \;:\; \overbrace{f_1(u,u) = f_2(u,u)}^{\text{consistency condition}} \right\}$$

e.g.

$(z_2 - z_1, \quad 0)$

$(1, \quad z_1 - z_2 + 1)$

$(2z_1^2, \quad z_1 z_2 + z_2^2)$

multiplication defined componentwise

Ex $\quad k=1 \quad n=2$

$$H^*_T(Gr_1\,\mathbb{C}^2) =$$

$$\Big\{ \big(f_1(z_1,z_2),\, f_2(z_1,z_2)\big) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \;:\; \overbrace{\underbrace{f_1(u,u) = f_2(u,u)}}^{\text{consistency condition}} \Big\}$$

equivalently: $\quad z_1 - z_2 \mid f_1(z_1,z_2) - f_2(z_1,z_2)$

$(z_2 - z_1,\quad 0)$	$f_1 - f_2 = z_2 - z_1$
$(1,\quad z_1 - z_2 + 1)$	$z_2 - z_1$
$(2z_1^2,\quad z_1 z_2 + z_2^2)$	$2z_1 - z_1 z_2 - z_2^2 = (z_1 - z_2)(2z_1 + z_2)$

Ex $k=1$ $n=2$

$$H^*_T(Gr_1 \mathbb{C}^2) =$$

$$\left\{ (f_1(z_1,z_2), f_2(z_1,z_2)) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \ : \ \overbrace{f_1(u,u) = f_2(u,u)}^{\text{consistency condition}} \right\}$$

$$z_1 - z_2 \mid f_1 - f_2$$

another rephrasing:

$\exists f(t, z_1, z_2)$ such that
$$f_1(z_1,z_2) = f(z_1, z_1, z_2)$$
$$f_2(z_1,z_2) = f(z_2, z_1, z_2)$$

$(z_2 - z_1, \quad 0)$ $\qquad f = z_2 - t$

$(1, \quad z_1 - z_2 + 1)$ $\qquad f = z_1 - t + 1$

$(2z_1^2, \quad z_1 z_2 + z_2^2)$ $\qquad f = t^2 + t z_1$

Ex $k=1 \quad n=2$

$$H_T^*(Gr_1 \mathbb{C}^2) =$$

$$\{ (f_1(z_1,z_2), f_2(z_1,z_2)) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \ : \ f_1(u,u) = f_2(u,u) \}$$

$$z_1 - z_2 \mid f_1 - f_2$$

$\exists f(t, z_1, z_2)$ s.t.

$$f_1 = f(z_1, z_1, z_2)$$
$$f_2 = f(z_2, z_1, z_2)$$

three equivalent ways of phrasing the consistency condition

Ex $k=2 \quad n=4$

$$H^*_T(Gr_2 \mathbb{C}^4) = \left\{ (f_{12}, f_{13}, \ldots, f_{34}) \in \mathbb{Z}[z_1, z_2, z_3, z_4]^6 : \text{consistency condition} \right\}$$

2-element subsets of $\{1,2,3,4\}$

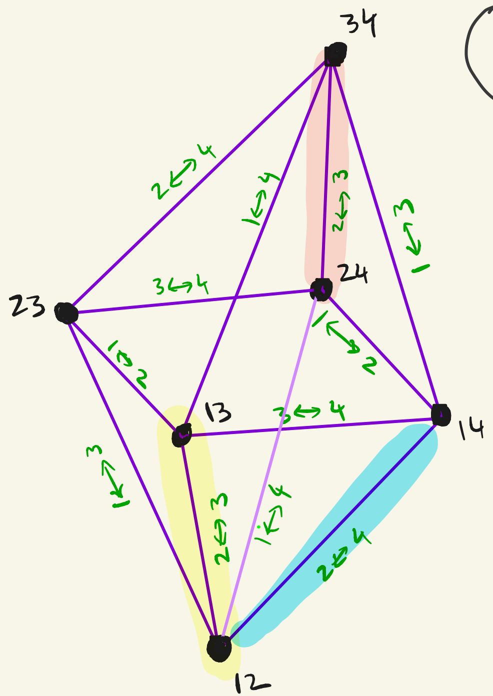
...

three equivalent ways of phrasing

⋮

Ex $k=2$ $n=4$

$$H_T^*(Gr_2\mathbb{C}^4) = \Big\{ \underbrace{(f_{12}, f_{13}, \dots, f_{34})}_{\text{six vertices}} \in \mathbb{Z}[z_1, z_2, z_3, z_4]^6 : \underbrace{\left\{ \begin{array}{l|l} z_2 - z_3 & f_{12} - f_{13} \\ z_2 - z_4 & f_{12} - f_{14} \\ \vdots & \\ z_2 - z_3 & f_{24} - f_{34} \end{array} \right.}_{\text{twelve edges}} \Big\}$$



<u>rem</u> say, $z_2 - z_3 \mid f_{12} - f_{13}$ is equivalent

$$f_{12}(z_1, u, u, z_4) = f_{13}(z_1, u, u, z_4)$$

$$\begin{array}{cccccc} f_{12} & f_{13} & f_{14} & f_{23} & f_{24} & f_{34} \\ \end{array}$$

$$\Big((z_4-z_1)(z_4-z_2),\quad (z_4-z_1)(z_4-z_3),\quad 0,\quad (z_4-z_2)(z_4-z_3),\quad 0,\quad 0 \Big)$$

= @ $z_2=z_3$

equal @ $z_1=z_3$

equal@ $z_1=z_4$

equal @ $z_1=z_4$

octahedron edges

Third equivalent rephrasing of consistency condition

$$H_T^*(Gr_2 \mathbb{C}^4) = \Big\{ (f_{12}, \dots, f_{34}) \in \mathbb{Z}[z_1, z_2, z_3, z_4]^6 :$$

$$\exists\, f(t_1, t_2, z_1, z_2, z_3, z_4) \in \mathbb{Z}[\overbrace{t_1, t_2}, z_1, z_2, z_3, z_4]^{S_2} \text{ such that}$$

$$f_{12} = f(z_1, z_2, z_1, z_2, z_3, z_4)$$

$$f_{13} = f(z_1, z_3, z_1, z_2, z_3, z_4)$$

$$\vdots$$

$$f_{34} = f(z_3, z_4, z_1, z_2, z_3, z_4) \Big\}$$

General $k \leq n$.

$$H_T^*(\mathrm{Gr}_k \mathbb{C}^n) = \left\{ (f_I) \in \mathbb{Z}[z_1, \dots, z_n]^{\binom{n}{k}} : \text{consistency} \right\}$$

k-element subset of $\{1 \dots n\}$

General $k \le n$.

$$H^*_T(Gr_k \mathbb{C}^n) = \left\{ (f_I) \in \mathbb{Z}[z_1, \dots, z_n]^{\binom{n}{k}} \; : \; \text{consistency} \right\}$$

$\forall\, I, J$ satisfying $I = K \cup \{i\}$
$J = K \cup \{j\}$

$z_i - z_j \mid f_I - f_J$

$\exists f \in \mathbb{Z}[\overbrace{t_1, \dots, t_k}, z_1, \dots, z_n]^{S_k}$

$$f_I = f(z_I, z_1, \dots, z_n)$$

So far:

$$\boxed{H^*_T(Gr_k\mathbb{C}^n) = \text{explicit description}}$$

enough for most of this lecture

Grassmannians

$0 \leq k \leq n$ integers

$$\mathrm{Gr}_k \mathbb{C}^n := \{ V^k \leq \mathbb{C}^n \} \qquad \mathrm{Gr}_1 \mathbb{C}^n =: \mathbb{P}^{n-1}$$

<u>Geometry</u>

- torus action, fix pts
- bundles over $\mathrm{Gr}_k \mathbb{C}^n$
- Schubert decomposition

torus action on $Gr_k\mathbb{C}^n$

$$\underbrace{(\mathbb{C}^*)^n}_{\text{torus} = T^n = T} \circlearrowright \mathbb{C}^n \quad \text{by} \quad (\zeta_1, \dots, \zeta_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \zeta_1 x_1 \\ \zeta_2 x_2 \\ \vdots \\ \zeta_n x_n \end{pmatrix}$$

<u>induces</u> $(\mathbb{C}^*)^n \circlearrowright Gr_k\mathbb{C}^n$ by $\zeta \cdot V^k = \{\zeta x : x \in V^k\}$

<u>fixed points</u>: coordinate k-planes $\overset{1:1}{\longleftrightarrow}$ k-element subsets of $\{1 \dots n\}$

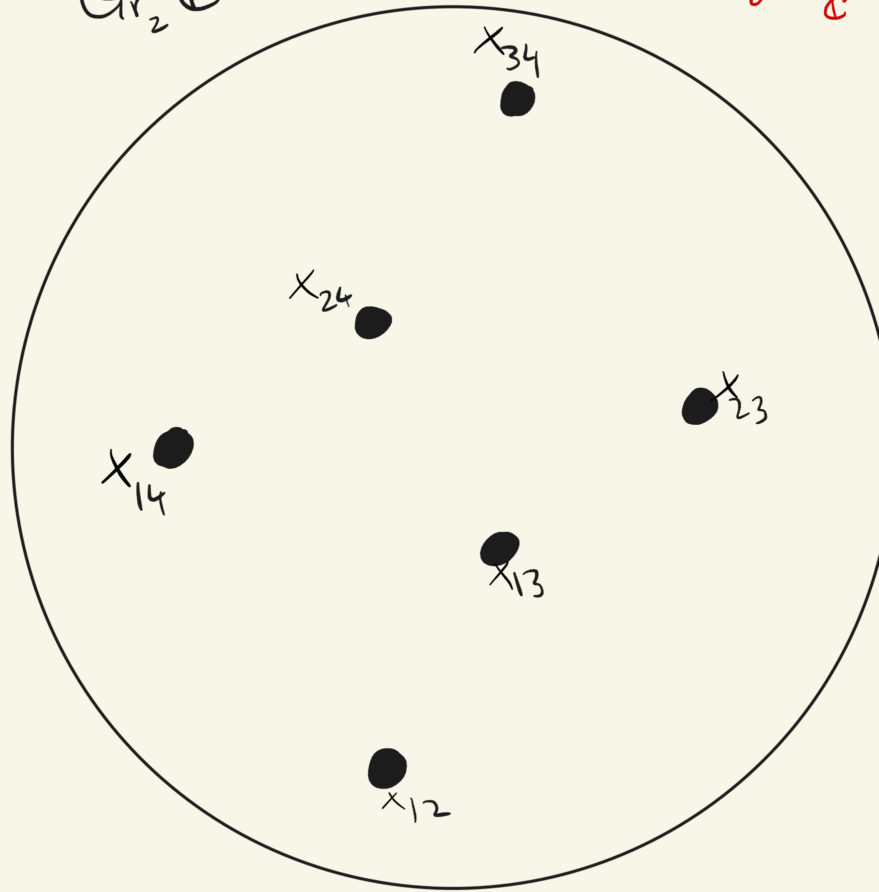
$$x_I \longleftrightarrow I$$

$\mathrm{Gr}_1\,\mathbb{C}^2 = \mathbb{P}^1$

$\dim_{\mathbb{C}} = 1$

$\mathrm{Gr}_2\,\mathbb{C}^4$

$\dim_{\mathbb{C}} = 4$



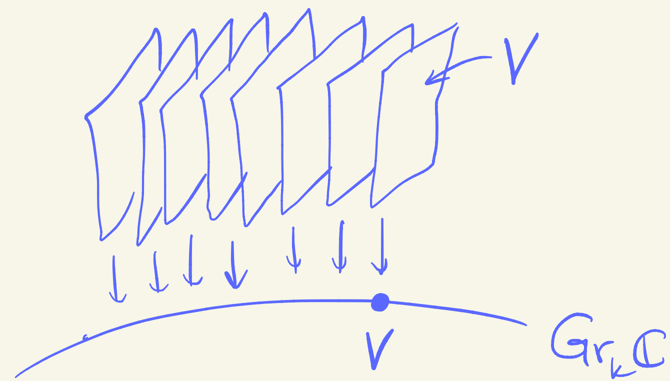
Tautological bundle over $Gr_k\mathbb{C}^n$

$$\mathcal{S} = \{ (V, x) \in Gr_k\mathbb{C}^n \times \mathbb{C}^n \ : \ x \in V \}$$

$T^n C$

$$\begin{array}{c} \mathcal{S} \\ \Big\downarrow \mathbb{C}^k \\ Gr_k\mathbb{C}^n \end{array}$$

" preimage of V is V "



Role: bundles determine elements in the cohomology of the base

"Chern class (bundle)"

Facts

$$\mathbb{Z}[\overbrace{t_1,\dots,t_k}, z_1 \dots z_n]^{S_k} \xrightarrow{q} H_T^*(Gr_k \mathbb{C}^n) \overset{Loc}{\rightarrowtail} \bigoplus_{\substack{|I|=k \\ I \subseteq \{1\dots n\}}} \underbrace{H_T^*(X_I)}_{\mathbb{Z}[z_1,\dots,z_n]}$$

- Loc = restrictions to fixed pts
  Loc injective

  and $\mathrm{im}(Loc) = \{ (f_I) : z_i - z_j \mid f_I - f_J \}$

  $I = K \cup \{i\}$
  $J = K \cup \{j\}$

- $t_1, \dots, t_k$ are Chern roots of S
  q surjective

- $$(Loc \circ q)_I = \begin{cases} t_1 \longmapsto z_{i_1} \\ t_2 \longmapsto z_{i_2} \\ \vdots \\ t_k \longmapsto z_{i_k} \end{cases} \qquad I = \{i_1, \dots, i_k\}$$

Where are we so far

$\mathrm{Gr}_k\,\mathbb{C}^n$

- torus action
- bundle

$\longrightarrow H_T^*(\mathrm{Gr}_k\,\mathbb{C}^n)$

explicit description

- Schubert decomposition

coming up

Schubert cells

$Gr_2 \mathbb{C}^4$

fix $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ "reference flag"

$$
\begin{array}{llcccc}
\Omega_{12} = \{ V : & \dim(V \cap \mathbb{C}^1) = 1, & \dim(V \cap \mathbb{C}^2) = 2, & \dim(V \cap \mathbb{C}^3) = 2, & \dim(V \cap \mathbb{C}^4) = 2 \} \\
\Omega_{13} = \{ V : & 1 & 1 & 2 & 2 \} \\
\Omega_{14} = \{ V : & 1 & 1 & 1 & 2 \} \\
\Omega_{23} = \{ V : & 0 & 1 & 2 & 2 \} \\
\Omega_{24} = \{ V : & 0 & 1 & 1 & 2 \} \\
\Omega_{34} = \{ V : & 0 & 0 & 1 & 2 \}
\end{array}
$$

$$Gr_2 \mathbb{C}^4 = \Omega_{12} \cup \Omega_{13} \cup \Omega_{14} \cup \Omega_{23} \cup \Omega_{24} \cup \Omega_{34}$$

$$Gr_2\mathbb{C}^4 = \Omega_{12} \cup \Omega_{13} \cup \Omega_{14} \cup \Omega_{23} \cup \Omega_{24} \cup \Omega_{34}$$

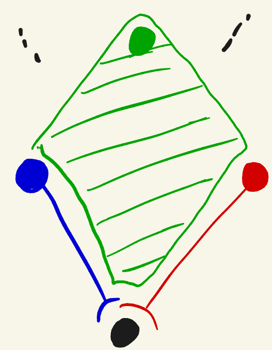
Ω_{12}: pt
Ω_{13}: dim $=1$
Ω_{14}: dim $=2$
Ω_{23}: dim $=2$
Ω_{24}: dim $=3$
Ω_{34}: dim $=4$

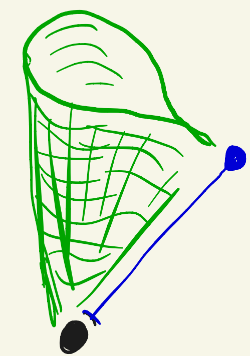
- Ω_{ij} cell $\cong \mathbb{C}$[redacted]

  Schubert cell

- $\overline{\Omega}_{ij}$ Schubert variety

  in general not smooth





Schubert decomposition

fix $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^n$
"reference full flag"

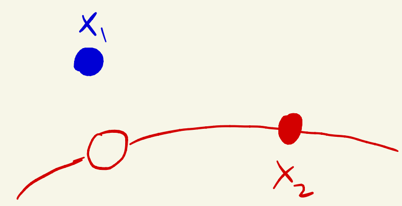
$I \subset \{1 \dots n\}$
$|I| = k$

$x_I \in$ $\quad \Omega_I := \{ V \in Gr_k \mathbb{C}^n \;:\; \dim(V \cap \mathbb{C}^q) = |\{i \in I : i \leq q\}| \}$

$\mathbb{P}^1$

$\Omega_1 =$

$\Omega_2 =$



Schubert decomposition induces a partial order on

Schubert-cells $\longleftrightarrow$ fixpts $\longleftrightarrow$ k-element subsets of $\{1 \dots n\}$

$$\Omega_I \geq \Omega_J$$

if

$$\overline{\Omega_I} \supseteq \Omega_J$$

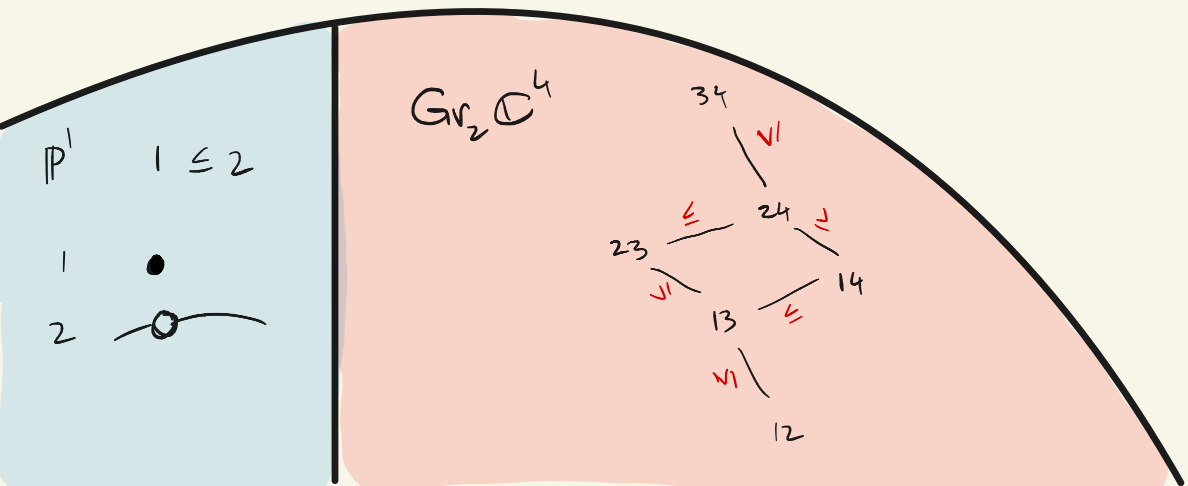
$$I = \{i_1 \leq \dots \leq i_k\}$$

$$J = \{j_1 \leq \dots \leq j_k\}$$

$$I \geq J$$

if

$$i_r \geq j_r \quad \forall r$$



Where are we so far

$\mathrm{Gr}_k\mathbb{C}^n$

- torus action
- bundle
- Schubert cells

$\rightarrow H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$ explicit description

Coming up

Schubert cell $\longrightarrow$ $\boxed{\text{"its class"}}$ $\in H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$

two versions:

- $[\overline{\Omega}_I]$ Schubert class
- $c^{sm}(\Omega_I)$ CSM class

Schubert classes

	12	13	14	23	24	34
$[\bar{\Omega}_{12}]$	$(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)$	0	0	0	0	0
$[\bar{\Omega}_{13}]$	----	$(z_2-z_1)(z_4-z_1)(z_4-z_3)$	0	0	0	0
$[\bar{\Omega}_{14}]$	----	----	$(z_2-z_1)(z_3-z_1)$	0	0	0
$[\bar{\Omega}_{23}]$	----	----	0	$(z_4-z_2)(z_4-z_3)$	0	0
$[\bar{\Omega}_{24}]$	----	----	----	----	(z_3-z_2)	0
$[\bar{\Omega}_{34}]$	----	----	----	----	----	1

Schubert classes

Thm-Def

$[\bar{\Omega}_I]$ is the unique class in $H^*_T(Gr_k \mathbb{C}^n)$

- degree $= \#\{(i,j) : i \in I,\ j \in \bar{I},\ j > i\}$

- $$[\bar{\Omega}_I]\big|_I = \prod_{\substack{i \in I \\ j \in \bar{I} \\ j > i}} (z_j - z_i)$$

$J \leq I$

$J = \{ j_1 < j_2 < \dots < j_k \}$

$\quad \ \geq \quad \geq \quad \dots \quad \geq$

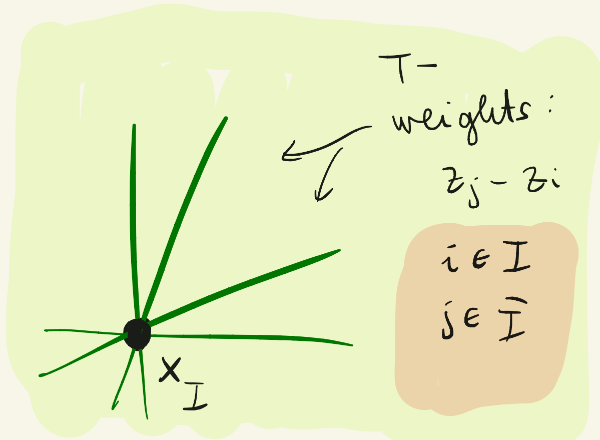
$I = \{ i_1 < i_2 < \dots < i_k \}$

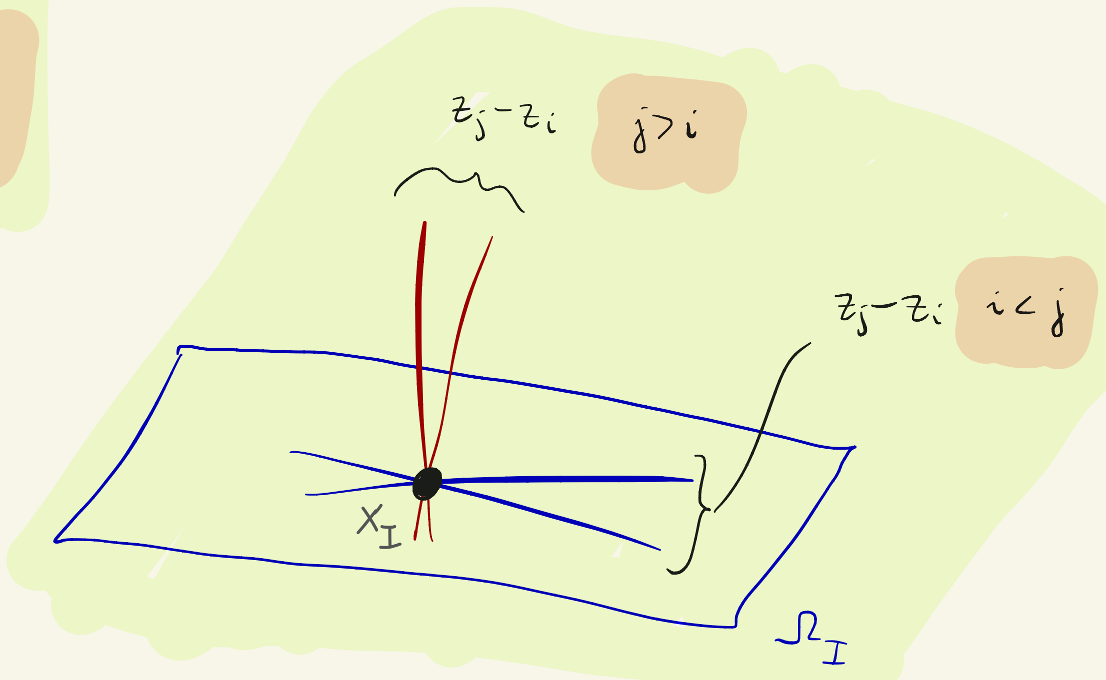
- $[\bar{\Omega}_I]\big|_J = 0$ if $J \not\leq I$

"axiomatic characterization of Schubert classes"

	12	13	14	23	24	34
$[\bar{\Omega}_{12}]$	$(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)$	0	0	0	0	0
$[\bar{\Omega}_{13}]$	$(z_3-z_1)(z_4-z_1)(z_4-z_2)$	$(z_2-z_1)(z_4-z_1)(z_4-z_3)$	0	0	0	0
$[\bar{\Omega}_{14}]$	$(z_3-z_1)(z_4-z_1)$	$(z_2-z_1)(z_4-z_1)$	$(z_2-z_1)(z_3-z_1)$	0	0	0
$[\bar{\Omega}_{23}]$	$(z_4-z_1)(z_4-z_2)$	$(z_4-z_1)(z_4-z_3)$	0	$(z_4-z_2)(z_4-z_3)$	0	0
$[\bar{\Omega}_{24}]$	$z_4+z_3-z_1-z_2$	z_4-z_1	z_3-z_1	z_4-z_2	(z_3-z_2)	0
$[\bar{\Omega}_{34}]$	1	1	1	1	1	1

Geometric explanation of diagonal restrictions





$Gr_k \mathbb{C}^n$ around fixpoint x_I

recall

$$H_T^*(Gr_2\mathbb{C}^4) = \begin{cases} \{ (f_{12}, \dots, f_{34}) : \quad \dots \quad \} \\ \{ (f_{12}, \dots, f_{34}) : \quad \exists f(t_1, t_2, \dots) \quad \dots \quad \} \end{cases}$$

	12	13	14	23	24	34
$[\Omega_{12}]$	$(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)$	0	0	0	0	0

$t_1 = z_1, \ t_2 = z_2$

$t_1 = z_1, \ t_2 = z_3$

$t_1 = z_3 \quad t_2 = z_4$

$$(z_3 - t_1)(z_4 - t_1)(z_3 - t_2)(z_4 - t_2)$$

	12	13	14	23	24	34
$[\bar{\Omega}_{13}]$	$(z_3-z_1)(z_4-z_1)(z_4-z_2)$	$(z_2-z_1)(z_4-z_1)(z_4-z_3)$	0	0	0	0
	$t_1=z_1$ $t_2=z_2$	$t_1=z_1$ $t_2=z_3$	$t_1=z_1$ $t_2=z_4$			$t_1=z_3$ $t_2=z_4$

$$\frac{(z_2-t_1)(z_3-t_1)(z_4-t_1)(z_4-t_2)}{(t_2-t_1)} + \frac{(z_2-t_2)(z_3-t_2)(z_4-t_2)(z_4-t_1)}{(t_1-t_2)}$$

$t_1 \leftrightarrow t_2$

- polynomial (easy algebra) (concrete simplified form is not important)

In general, $Gr_k\mathbb{C}^n$, $I = \{i_1 < i_2 < \dots < i_k\}$

$$[\bar\Omega_I] = \operatorname*{Sym}_{t_1,\dots,t_k} \left(\prod_{a=1}^{k} \prod_{b=i_a+1}^{n} (z_b - t_a) \cdot \prod_{1\le a<b\le k} \frac{1}{t_b - t_a} \right)$$

- polynomial
- satisfies the axioms

$$[\bar\Omega_I]\Big|_I = \prod_{\substack{i\in I\\ j\in \bar I\\ j>i}} (z_j - z_i)$$

$$[\bar\Omega_I]\Big|_J = 0 \quad \text{if} \quad J \notin I$$

Rem substituting $z_i = 0$ the formula simplifies to

$$\det \begin{pmatrix} t_1^{n-i_1} & t_2^{n-i_1} & t_3^{n-i_1} & \cdots \\ t_1^{n-i_2} & t_2^{n-i_2} & t_3^{n-i_2} & \cdots \\ \vdots & & & \end{pmatrix} \Bigg/ \det \begin{pmatrix} 1 & t_1 & t_1^2 & \\ 1 & t_2 & t_2^2 & \\ 1 & t_3 & t_3^2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Schur polynomial

Where are we so far

- $H^*_T(Gr_k\mathbb{C}^n)$ description $= \{ (f_I) : \ldots \}$
- $[\bar{\Omega}_I] \in$

defined by interpolation axioms

by $f(t_1 \ldots t_k, z_1 \ldots z_n)$ formula

coming up

$\hbar$-deformation of $[\bar{\Omega}_I]$

CSM classes

"Chern-Schwartz-MacPherson"

($\hbar$-deformed cohomological Schubert classes)

	12	13	14	23	24	34
$c^{sm}(\Omega_{12})$	$\boxed{(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)}$	0	0	0	0	0
$c^{sm}(\Omega_{13})$	$\hbar \cdot \ldots$	$\boxed{(z_2-z_1)(z_4-z_1)(z_4-z_3)(z_2-z_3+\hbar)}$	0	0	0	0
$c^{sm}(\Omega_{14})$	$\hbar \cdot \ldots$	$\hbar \cdot (z_2-z_3+\hbar) \cdot \ldots$	$\boxed{(z_2-z_1)(z_3-z_1)(z_2-z_4+\hbar)(z_3-z_4+\hbar)}$	0	0	0
$c^{sm}(\Omega_{23})$	$\hbar \cdot \ldots$	$\hbar \cdot (z_2-z_3+\hbar) \cdot \ldots$	0	$\boxed{(z_4-z_2)(z_4-z_3)(z_1-z_2+\hbar)(z_1-z_3+\hbar)}$	0	0
$c^{sm}(\Omega_{24})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar) \cdot \ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar) \cdot \ldots$	$\hbar(\ldots)(\ldots) \ldots$	$\boxed{(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_4+\hbar)(z_3-z_4+\hbar)}$	0
$c^{sm}(\Omega_{34})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar) \cdot \ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar) \cdot \ldots$	$\hbar(\ldots)(\ldots) \ldots$	$\hbar(\cdot)(\cdot)(\cdot) \ldots$	$\boxed{(z_1-z_3+\hbar)(z_1-z_4+\hbar)(z_2-z_3+\hbar)(z_2-z_4+\hbar)}$

$c^{sm}(\Omega_I)$ is the unique class in $H_T^*(Gr_k \mathbb{C}^n)[\hbar]$

- degree = dim (Gr) (deg $\hbar = 1$)
- $$c^{sm}(\Omega_I)\Big|_I = \prod_{\substack{i \in I \\ j \in \bar{I} \\ i<j}} (z_j - z_i) \cdot \underbrace{\prod_{\substack{i \in I \\ j \in \bar{I} \\ i>j}} (z_j - z_i + \hbar)}_{c_I}$$
- $c^{sm}(\Omega_I)\big|_J$ divisible by $\hbar$ for $I \neq J$
- $c^{sm}(\Omega_I)\big|_J$ divisible by c_J

(• $c^{sm}(\Omega_I)\big|_J = 0$ if $J \not\leq I$)

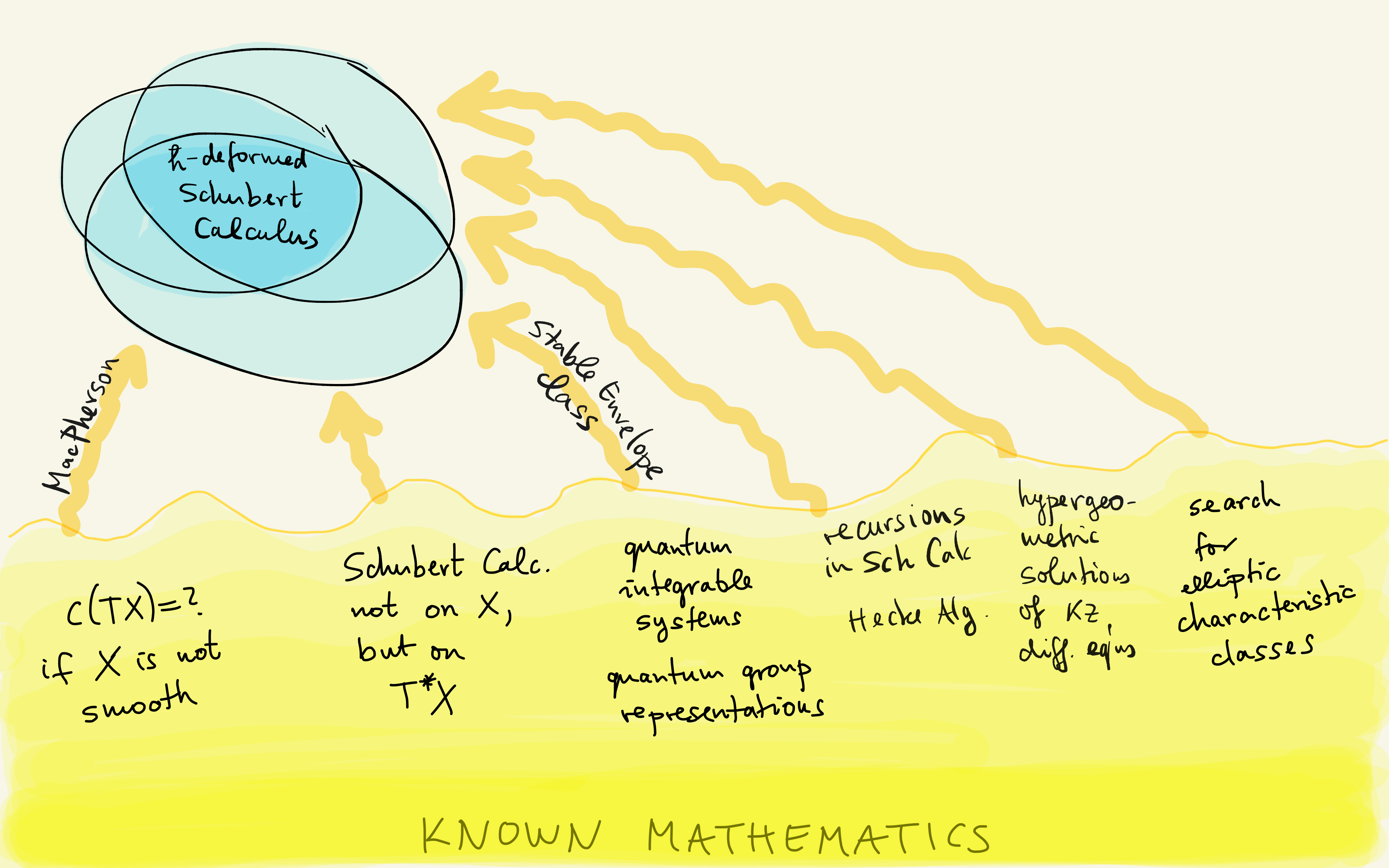
	12	13	14	23	24	34
$c^{sm}(\Omega_{12})$	$\boxed{(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)}$	0	0	0	0	0
$c^{sm}(\Omega_{13})$	$\hbar(z_3-z_1)(z_4-z_1)(z_4-z_2)$	$\boxed{(z_2-z_1)(z_4-z_1)(z_4-z_3)(z_2-z_3+\hbar)}$	0	0	0	0
$c^{sm}(\Omega_{14})$	$\hbar\cdot(z_3-z_1)(z_4-z_1)(z_3-z_2+\hbar)$	$\hbar\cdot(z_2-z_3+\hbar)(z_2-z_1)(z_4-z_1)$	$\boxed{(z_2-z_1)(z_3-z_1)(z_2-z_4+\hbar)(z_3-z_4+\hbar)}$	0	0	0
$c^{sm}(\Omega_{23})$	$\hbar\cdot(\ )(\ )(\ )$	$\hbar\cdot(z_2-z_3+\hbar)\cdot(--)(\ )$	0	$\boxed{(z_4-z_2)(z_4-z_3)(z_1-z_2+\hbar)(z_1-z_3+\hbar)}$	0	0
$c^{sm}(\Omega_{24})$	$\hbar\cdot\boxed{\text{MESS}}$	$\hbar(z_2-z_3+\hbar)(\ )(\ )$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)(\ )$	$\hbar(--)(--)(\ )$	$\boxed{(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_4+\hbar)(z_3-z_4+\hbar)}$	0
$c^{sm}(\Omega_{34})$	$\hbar\cdot\boxed{\text{MESS}}$	$\hbar(z_2-z_3+\hbar)(\ )(\ )$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)(\cdot)$	$\hbar(--)(-)(\ )$	$\hbar(\cdot)(\cdot)(--)\ldots$	$\boxed{(z_1-z_3+\hbar)(z_1-z_4+\hbar)(z_2-z_3+\hbar)(z_2-z_4+\hbar)}$

$\mathbb{P}^1$

	1	2	
$c^{sm}(\Omega_1)$	$(z_2 - z_1)$	0	$= z_2 - t$
$c^{sm}(\Omega_2)$	$?$	$(z_1 - z_2 + \hbar)$	$= z_1 - t + \hbar$

$$\hbar \cdot (\deg\text{-}0) = \hbar \cdot A$$

$$\hbar A \underset{\text{when } z_1 = z_2}{=} z_1 - z_2 + \hbar \implies A = 1$$


ℏ-deformed Schubert Calculus
MacPherson
Stable Envelope class
c(TX)=?
if X is not smooth
Schubert Calc. not on X, but on T*X
quantum integrable systems
quantum group representations
recursions in Sch Calc
Hecke Alg.
hypergeo-metric solutions of KZ diff. eq'ns
search for elliptic characteristic classes
KNOWN MATHEMATICS

Corrected vocab:

- what we called $c^{sm}(\Omega_I)$ is in fact $\mathrm{Stab}(\Omega_I)$

- $c^{sm}(\Omega_I) = \mathrm{Stab}(\Omega_I)\Big|_{\hbar=1}$

(same information, one determines the other)

Where are we?

$$\mathrm{Gr}_k\mathbb{C}^n \quad \begin{cases} \text{torus action} \\ \text{bundle} \\ \text{Schubert cells} \end{cases}$$

$\left.\begin{array}{l}\text{torus action}\\ \text{bundle}\end{array}\right\} \to H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$

Schubert cells $\downarrow$; $H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$ $\downarrow$

Schubert class $\qquad [\bar{\Omega}_I] \in H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$

$\left.\begin{array}{l}\hbar\text{-deformed Schubert class}\\ \text{CSM class } (\hbar=1)\\ \text{stable envelope class}\end{array}\right\} \quad c^{sm}(\Omega_I) \in H_T^*(\mathrm{Gr}_k\mathbb{C}^n)$

Rest of the talk:

illustrations of some properties of c^{sm} classes

- R-matrix property ... Yang-Baxter equation
- MacPherson point-of-view ... $c^{sm}(A \cup B)$
  $c^{sm}(\text{smooth compact})$
- formulas ... weight functions ... Schur expansion
- why "cotangent" Schubert Calculus
- recursions
- how is c^{sm} "enumerative geometry"
- K theory version

R-matrix

Basis of $H^*_{T^2}\left(Gr_0\mathbb{C}^2 \sqcup Gr_1\mathbb{C}^2 \sqcup Gr_2\mathbb{C}^2\right)$

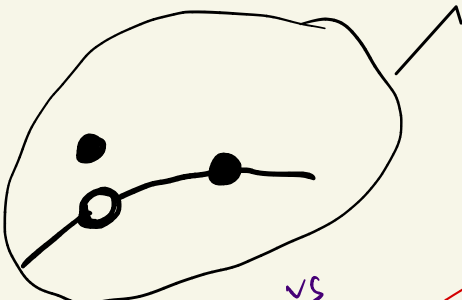
$$c^{sm}\left(\Omega_\emptyset \subset Gr_0\mathbb{C}^2\right) = 1 \qquad \in H^*_T\left(Gr_0\mathbb{C}^2\right)$$

$$\left.\begin{aligned} c^{sm}\left(\Omega_1 \subset Gr_1\mathbb{C}^2\right) &= z_2 - t \\ c^{sm}\left(\Omega_2 \subset Gr_1\mathbb{C}^2\right) &= z_1 - t + \hbar \end{aligned}\right\} \in H^*_T\left(Gr_1\mathbb{C}^2\right)$$

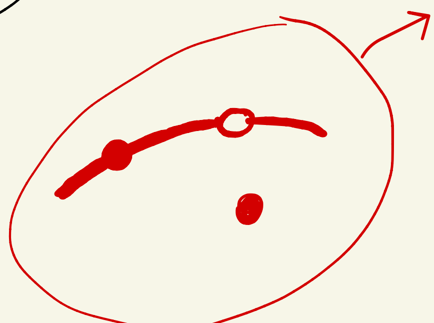
$$c^{sm}\left(\Omega_{12} \subset Gr_2\mathbb{C}^2\right) = 1 \qquad \in H^*_T\left(Gr_2\mathbb{C}^2\right)$$

$$\boxed{\begin{aligned}
c^{sm}(\Omega_\emptyset \subset Gr_0\mathbb{C}^2) &= 1\\
c^{sm}(\Omega_1 \subset Gr_1\mathbb{C}^2) &= z_2 - t\\
c^{sm}(\Omega_2 \subset Gr_1\mathbb{C}^2) &= z_1 - t + \hbar\\
c^{sm}(\Omega_{12} \subset Gr_2\mathbb{C}^2) &= 1
\end{aligned}}$$

csm classes with "opposite" reference flag



vs



$$\boxed{\begin{aligned}
c^{sm}(\Omega_\emptyset \subset Gr_0\mathbb{C}^2) &= 1\\
c^{sm}(\Omega_1 \subset Gr_1\mathbb{C}^2) &= z_2 - t + \hbar\\
c^{sm}(\Omega_2 \subset Gr_1\mathbb{C}^2) &= z_1 - t\\
c^{sm}(\Omega_{12} \subset Gr_2\mathbb{C}^2) &= 1
\end{aligned}}$$

Calculation:

change of basis matrix:

$$R(z_1 - z_2) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{z_1 - z_2}{z_1 - z_2 + \hbar} \mathbf{Id} + \frac{\hbar}{z_1 - z_2 + \hbar} \mathbf{P}$$

Fact this matrix satisfies the

parameter-dependent Yang-Baxter equation:

$$R_{12}(z_1-z_2)\,R_{13}(z_1-z_3)\,R_{23}(z_2-z_3) = R_{23}(z_2-z_3)\,R_{13}(z_1-z_3)\,R_{12}(z_1-z_2)$$

(verify!!!)

meaning:

$\mathbb{C}^2 = \mathrm{span}(v_1, v_2)$ $\otimes\mathbb{C}(z_1,z_2)$

$v_1\otimes v_1 \longleftrightarrow C^{sm}(\Omega_\emptyset)$

$v_1\otimes v_2 \longleftrightarrow C^{sm}(\Omega_1)$

$v_2\otimes v_1 \longleftrightarrow C^{sm}(\Omega_2)$

$v_2\otimes v_2 \longleftrightarrow C^{sm}(\Omega_{12})$

$$\boxed{\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2 \longrightarrow \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2}$$

$R_{ij}(z)$ acts as $R(z)$ in i'th & j'th factor

$$R_{12}(z_1-z_2)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_1-z_2}{z_1-z_2+h} & 0 & \frac{h}{z_1-z_2+h} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{z_1-z_2}{z_1-z_2+h} & 0 & \frac{h}{z_1-z_2+h} & 0 & 0 \\ 0 & 0 & \frac{h}{z_1-z_2+h} & 0 & \frac{z_1-z_2}{z_1-z_2+h} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h}{z_1-z_2+h} & 0 & \frac{z_1-z_2}{z_1-z_2+h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_2-z_3}{z_2-z_3+h} & \frac{h}{z_2-z_3+h} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h}{z_2-z_3+h} & \frac{z_2-z_3}{z_2-z_3+h} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{z_2-z_3}{z_2-z_3+h} & \frac{h}{z_2-z_3+h} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h}{z_2-z_3+h} & \frac{z_2-z_3}{z_2-z_3+h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R_{23}(z_2-z_3)$$

$$R_{13}(z_1-z_3)=\left[\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-z_3+z_1}{z_1-z_3+h} & 0 & 0 & \frac{h}{z_1-z_3+h} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-z_3+z_1}{z_1-z_3+h} & 0 & 0 & \frac{h}{z_1-z_3+h} & 0 \\ 0 & \frac{h}{z_1-z_3+h} & 0 & 0 & \frac{-z_3+z_1}{z_1-z_3+h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{h}{z_1-z_3+h} & 0 & 0 & \frac{-z_3+z_1}{z_1-z_3+h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right]\begin{matrix} v_1\otimes v_1\otimes v_1 \\ v_1\otimes v_1\otimes v_2 \\ v_1\otimes v_2\otimes v_1 \\ v_1\otimes v_2\otimes v_2 \\ 2\ \ 1\ \ 1 \\ 2\ \ 1\ \ 2 \\ 2\ \ 2\ \ 1 \\ 2\ \ 2\ \ 2 \end{matrix}$$

Further along this direction ...

this particular solution of the YB equation is the R-matrix of $Y(gl_2)$

Yangian

$\longrightarrow$ $H_T^*\left(\bigsqcup_k Gr_k\mathbb{C}^n\right)$ is a $Y(gl_2)$-module

(with c^{sm}-classes playing the role of standard basis vectors)

"spin basis"

MacPherson property of CSM classes

	12	13	14	23	24	34
$c^{sm}(\Omega_{12})$	$\boxed{(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)}$	0	0	0	0	0
$c^{sm}(\Omega_{13})$	$\hbar \cdot \ldots$	$\boxed{(z_2-z_1)(z_4-z_1)(z_4-z_3)(z_2-z_3+\hbar)}$	0	0	0	0
$c^{sm}(\Omega_{14})$	$\hbar \cdot \ldots$	$\hbar\cdot(z_2-z_3+\hbar)\cdot\ldots$	$\boxed{(z_2-z_1)(z_3-z_1)(z_2-z_4+\hbar)(z_3-z_4+\hbar)}$	0	0	0
$c^{sm}(\Omega_{23})$	$\hbar \cdot \ldots$	$\hbar\cdot(z_2-z_3+\hbar)\cdot\ldots$	0	$\boxed{(z_4-z_2)(z_4-z_3)(z_1-z_2+\hbar)(z_1-z_3+\hbar)}$	0	0
$c^{sm}(\Omega_{24})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar)\cdot\ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)\cdot\ldots$	$\hbar(\ldots)(\ldots)\ldots$	$\boxed{(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_4+\hbar)(z_3-z_4+\hbar)}$	0
$c^{sm}(\Omega_{34})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar)\cdot\ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)\cdot\ldots$	$\hbar(\ldots)(\ldots)\ldots$	$\hbar(\cdot)(\cdot)(\cdot)\ldots$	$\boxed{(z_1-z_3+\hbar)(z_1-z_4+\hbar)(z_2-z_3+\hbar)(z_2-z_4+\hbar)}$

Σ add together all

$$\sum_I c^{sm}(\Omega_I) = \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ \end{array}$$

$$\left(\begin{array}{c|c|c|c|c|c} \begin{matrix}(z_3-z_1+\hbar)(z_3-z_2+\hbar)\\(z_4-z_1+\hbar)(z_4-z_2+\hbar)\end{matrix} & \begin{matrix}(z_2-z_1+\hbar)(z_4-z_1+\hbar)\\(z_2-z_3+\hbar)(z_4-z_3+\hbar)\end{matrix} & \begin{matrix}(z_2-z_1+\hbar)(z_3-z_1+\hbar)\\(z_2-z_4+\hbar)(z_3-z_4+\hbar)\end{matrix} & \begin{matrix}(z_1-z_2+\hbar)(z_4-z_2+\hbar)\\(z_1-z_3+\hbar)(z_4-z_3+\hbar)\end{matrix} & \begin{matrix}(z_1-z_2+\hbar)(z_3-z_2+\hbar)\\(z_1-z_4+\hbar)(z_3-z_4+\hbar)\end{matrix} & \begin{matrix}(z_1-z_3+\hbar)(z_2-z_3+\hbar)\\(z_1-z_4+\hbar)(z_2-z_4+\hbar)\end{matrix} \end{array} \right)$$

$\Rightarrow$

at each fixed point I it is $\displaystyle\prod_{\substack{i\in I\\ j\in \bar{I}}} (z_j - z_i + \hbar)$

but $\quad c(TGr_2\mathbb{C}^4)\big|_I = \displaystyle\prod_{\substack{i\in I\\ j\in \bar{I}}} (1 + z_j - z_i)$

$$\Longrightarrow \quad \sum c^{sm}(\Omega_I) = c\Big(T\big(\overbrace{\bigsqcup_I \Omega_I}^{\text{smooth}}\big)\Big) \qquad (\hbar = 1)$$

$$c^{sm}(\Omega_{23}) = \quad \dots \qquad \boxed{(z_4 - z_2)(z_4 - z_3)(z_1 - z_2 + \hbar)(z_1 - z_3 + \hbar)} \qquad 0 \qquad 0$$

$$c^{sm}(\Omega_{24}) = \quad \dots \qquad \hbar(\dots)(\dots)\dots \qquad \boxed{(z_3 - z_2)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar)} \qquad 0$$

$$c^{sm}(\Omega_{34}) = \quad \dots \qquad \hbar(\dots)(\dots)\dots \qquad \hbar(\cdot)(\cdot)(\cdot)\dots \qquad \boxed{(z_1 - z_3 + \hbar)(z_1 - z_4 + \hbar)(z_2 - z_3 + \hbar)(z_2 - z_4 + \hbar)}$$

$$\left.\begin{array}{l} (z_3 - z_2)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar) \\ \hbar(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar) \end{array}\right\} + \quad = (z_3 - z_2 + \hbar)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar)$$

More generally

, if $\bigsqcup_J \Omega_J = M \overset{i}{\subseteq} Gr_k \mathbb{C}^n$,

smooth cpt

then

$$\boxed{\sum_J c^{sm}(\Omega_J) = i_*\big(c(TM) \big)}$$

→ overdetermined notion

→ generalize to different settings

→ different conventions ($\hbar$ variable or $\hbar = 1$)

→ different names :(

$\hbar$-deformed Schubert Calculus

MacPherson

$c(TX) = ?$ if X is not smooth

Schubert Calc. not on X, but on T^*X

quantum integrable systems

quantum group representations

general recursions in Schubert Calculus

Hecke algebras

search for elliptic characteristic classes

KNOWN MATHEMATICS

More generally: $c^{sm}(f) \in H_T^*(Gr_k \mathbb{C}^n)$ is defined

constructible (T-invariant) function

meaning: can be written as

$$\sum c_V \cdot \underbrace{\mathbb{1}_V}_{\text{indicator function of variety } V}$$

- additive

$$c^{sm}(f+g) = c^{sm}(f) + c^{sm}(g)$$
$$c^{sm}(\lambda f) = \lambda \cdot c^{sm}(f)$$

- if $f = \mathbb{1}_M$ compact smooth then $c^{sm}(f) = i_*(c(TM))$

$$c^{sm}(\Omega_I) = c^{sm}(\mathbb{1}_{\Omega_i})$$

Even more generally:

$$C_*^T : \underbrace{\mathcal{F}^T(-)}_{\text{functor of } T\text{-invariant constructible functions on } \mathbb{C} \text{ algebraic varieties}} \longrightarrow \underbrace{H_*^T(-)}_{T\text{-equivariant homology functor}}$$

[covariant, pushforward defined via χ]

unique natural transformation of functors satisfying

- $1\!\!1_M \mapsto c(TM) \cap \mu_M^T$

Why "cotangent" Schubert Calculus

$\mathbb{P}^1$

$$c^{sm}(\Omega_1) = (z_2 - z_1, \quad 0)$$

$= [\{x_1\}]$ fundamental class

$$c^{sm}(\Omega_2) = (\hbar, \quad z_1 - z_2 + \hbar)$$

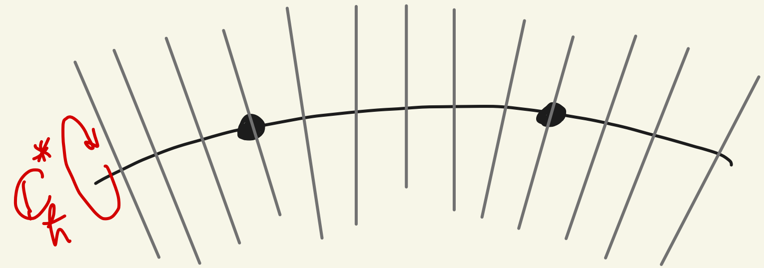
$\neq$ fundamental class

⋮

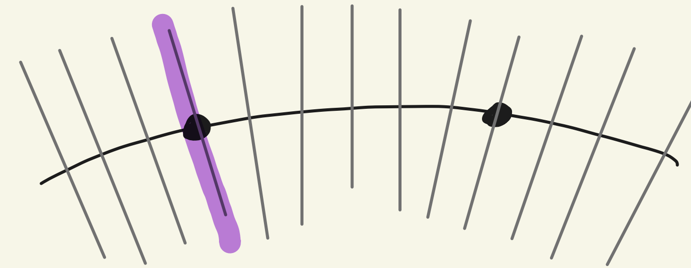
<u>but</u>

⋮

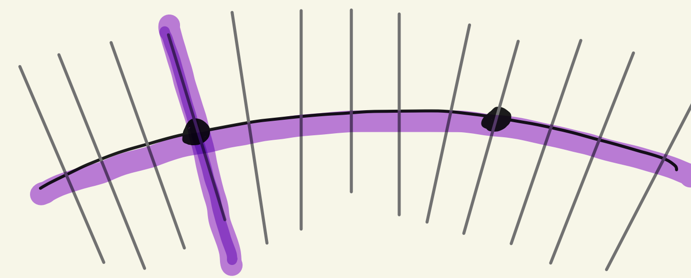
- replace $\mathbb{P}^1$ with $T^*\mathbb{P}^1$
  & act by $\mathbb{C}^*_\hbar$ in fibers

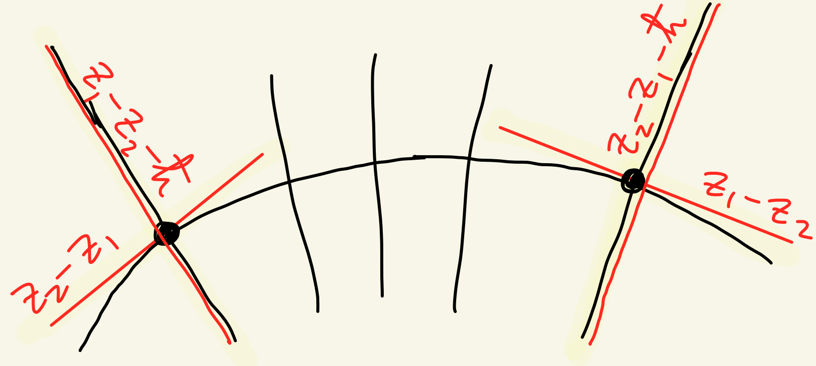


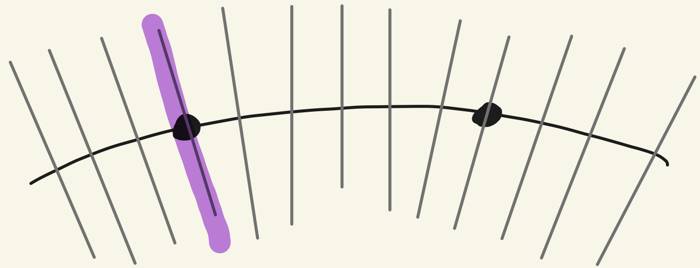
- consider these cycles

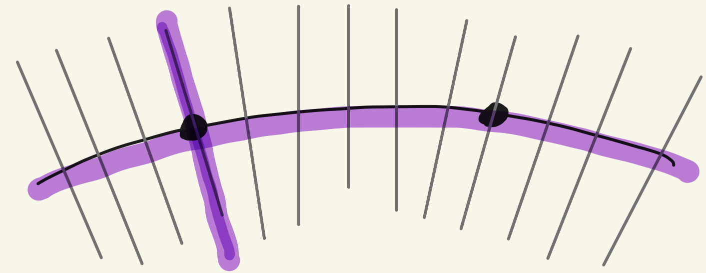


calculate their fundamental classes









[] $= (z_2 - z_1,\ 0)$

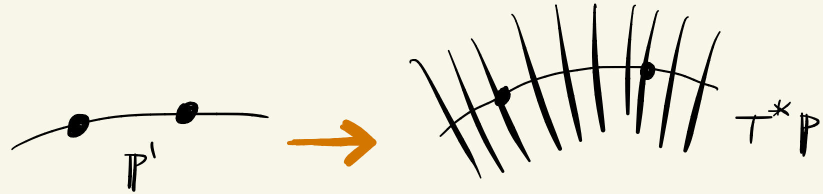
[] $= \big((z_2 - z_1) + (z_1 - z_2 - \hbar),\ z_2 - z_1 - \hbar \big)$

$= -(\hbar,\ z_1 - z_2 + \hbar)$

Summary

$X \longrightarrow T^*X$

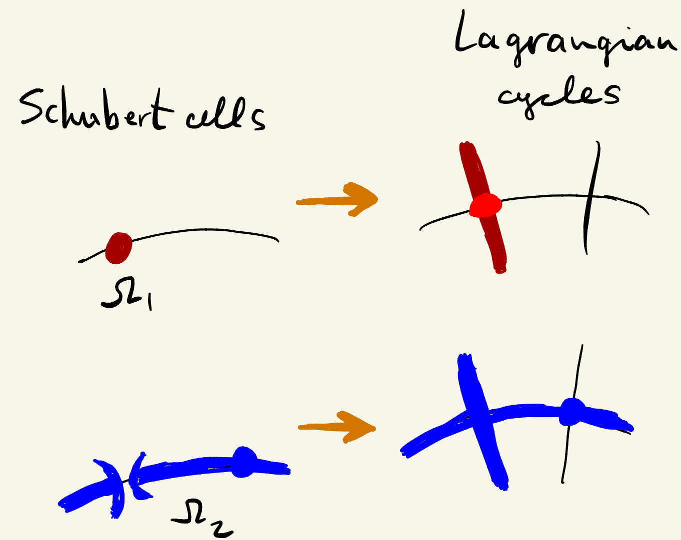
with extra $\mathbb{C}^*_\hbar$ action



$\Omega \longrightarrow$ "associated Lagrangian cycle $\subset T^*X$"

$\Omega_w \to \coprod_{\eta \leq w} c_{w\eta} \cdot C\Omega_\eta$

conormal bundle of Schubert cell



$\Longrightarrow$ $c^{sm} \longrightarrow$ fundamental class

Recursions

(c.f. Lascoux-Schützenberger recursion in Schubert Calculus [lecture yesterday])

$\mathcal{F}(3)$	123	132	213	231	312	321
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	0	0
$c^{sm}(\Omega_{132})$						
$c^{sm}(\Omega_{213})$						
$c^{sm}(\Omega_{231})$						
$c^{sm}(\Omega_{312})$						
$c^{sm}(\Omega_{321})$						

on full flag variety $= \{ \ell^1 \subseteq V^2 \subseteq \mathbb{C}^3 \}$

T-fixpoints $\leftrightarrow$ Schubert cells

$\leftrightarrow$ permutations

X = full flag variety $\qquad s_k = (k\ k{+}1) \in S_n$

Bott-Samelson recursion

$$c^{sm}(\Omega_{ws_k})\Big|_\sigma = \frac{\hbar}{z_{\sigma(k+1)} - z_{\sigma(k)}}\, c^{sm}(\Omega_w)\Big|_\sigma - \frac{z_{\sigma(k+1)} - z_{\sigma(k)} + \hbar}{z_{\sigma(k+1)} - z_{\sigma(k)}}\, c^{sm}(\Omega_w)\Big|_{\sigma s_k}$$

$\forall w, \sigma$

only for full flag varieties

R-matrix recursion

$$c^{sm}(\Omega_{s_k w})\Big|_\sigma = \frac{\hbar}{z_{k+1} - z_k}\, c^{sm}(\Omega_w)\Big|_\sigma + \frac{z_k - z_{k+1} + \hbar}{z_k - z_{k+1}} \left[c^{sm}(\Omega_w)\Big|_{s_k\sigma} \right]_{z_k \leftrightarrow z_{k+1}}$$

$\forall w, \sigma$

also for partial flag varieties

$\mathfrak{S}(3)$	123	132	213	231	312	321
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	0	0
$c^{sm}(\Omega_{132})$						

$$c^{sm}(\Omega_{s_k w})\Big|_\sigma = \frac{\hbar}{z_{k+1}-z_k}\, c^{sm}(\Omega_w)\Big|_\sigma + \frac{z_k - z_{k+1} + \hbar}{z_k - z_{k+1}} \left[c^{sm}(\Omega_w)\Big|_{s_k \sigma} \right]_{z_k \leftrightarrow z_{k+1}}$$

$w = 123 \quad k=2$
$\sigma = 123$

$$c^{sm}(\Omega_{132})\Big|_{123} = \frac{\hbar}{z_3-z_2}(z_2-z_1)(z_3-z_1)(z_3-z_2) + \frac{z_2-z_3+\hbar}{z_2-z_3}\cdot 0 = \hbar(z_2-z_1)(z_3-z_1)$$

$w = 123 \quad k=2$
$\sigma = 132$

$$c^{sm}(\Omega_{132})\Big|_{132} = \frac{\hbar}{z_3-z_2}\cdot 0 + \frac{z_2-z_3+\hbar}{z_2-z_3}\Big[(z_2-z_1)(z_3-z_1)(z_3-z_2)\Big]_{z_2\leftrightarrow z_3} =$$

$$= (z_2-z_3+\hbar)(z_3-z_1)(z_2-z_1)$$

$\mathcal{J}(3)$	123	132	213	231	312 321 $\cdots\to$
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	$\cdots$
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0	$\cdots$
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0	$\cdots$
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	$\cdots$
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0	$\cdots$
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	$\boxed{\text{Homework}}$	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	$\cdots$

$$= \hbar(\hbar^2 + z_2 z_1 - z_1 z_3 - z_2^2 + z_2 z_3)$$

$F(3)$	123	132	213	231	312	321
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0		
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0		
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0		
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$		
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0		
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	Homework	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$		

Bott-Samelson $w=312$ $k=2$
$\sigma=123$

$$\frac{\hbar}{z_3-z_2}\,\hbar^2(z_3-z_1)-\frac{z_3-z_2+\hbar}{z_3-z_2}\,\hbar(z_2-z_1)(z_2-z_3+\hbar)$$

Bott-Samelson $w=231$ $k=1$
$\sigma=123$

$$\frac{\hbar}{z_2-z_1}\,\hbar^2(z_3-z_1)-\frac{(z_2-z_1+\hbar)}{z_2-z_1}\,(z_1-z_2+\hbar)(z_3-z_2)\hbar$$

$F(3)$	123	132	213	231	312 321 ..... $\to$
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0	
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0	
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0	
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	Homework	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	

R-matrix recursion

$w=312 \quad k=1$

$\sigma=123$

$$\frac{\hbar}{z_2-z_1}\,\hbar^2(z_3-z_1)+\frac{z_1-z_2+\hbar}{z_1-z_2}\Big[(z_3-z_1)(z_1-z_2+\hbar)\hbar\Big]_{z_1\leftrightarrow z_2}$$

R-matrix recursion

$w=231 \quad k=2$

$\sigma=123$

$$\frac{\hbar}{z_3-z_2}\,\hbar^2(z_3-z_1)+\frac{z_2-z_3+\hbar}{z_2-z_3}\Big[(z_3-z_1)\hbar(z_2-z_3+\hbar)\Big]_{z_2\leftrightarrow z_3}$$

Remarks

- Bott–Samelson recursion only for full flag varieties
- R-matrix recursion for G/P (partial flag var's) too
- both recursions can be phrased "globally" as well, on $c^{sm}(\Omega_w)$'s not on "local" classes $c^{sm}(\Omega_w)|_\sigma$
- either one can serve as a definition of $c^{sm}(\Omega_I)$ (together with the obvious $c^{sm}(\Omega_{id}) = c^{sm}(\text{point})$)
- $c^{sm}(\Omega_w)|_\sigma$ overdetermined $\rightarrow$ identities for $\boxed{\text{rational}}$ functions

"K-theoretic version"

"elliptic cohomology version"

$\boxed{\text{rational}}$
$\downarrow$
$\boxed{\text{trigonometric}}$
$\downarrow$
$\boxed{\text{elliptic}}$

- full flag variety having 2 "dual" recursions is an incarnation of the fact that $T^*\mathcal{F}(n) \xleftrightarrow[\text{symmetry}]{\text{3d mirror}} T^*\mathcal{F}(n)$

in general $X \xleftrightarrow[\text{symmetry}]{\text{3d mirror}} X'$

$\parallel$

their (elliptic) Schubert calculus "match" (in a complicated sense)

Formulas for c^{sm} classes

recall

$$[\bar{\Omega}_I] = \operatorname*{Sym}_{t_1 \dots t_k} \left(\prod_{a=1}^{k} \left(\qquad \prod_{b=i_a+1}^{n} (z_b - t_a) \right) \prod_{1 \le a \le b \le k} \frac{1}{(t_b - t_a) \qquad} \right)$$

essentially the Vandermonde $\frac{\det(\dots)}{\det(\dots)}$ formula for Schur functions

---

fact

$$c^{sm}(\Omega_I) = \operatorname*{Sym}_{t_1 \dots t_k} \left(\prod_{a=1}^{k} \left(\prod_{b=1}^{i_a - 1} (z_b - t_a + \hbar) \prod_{b=i_a+1}^{n} (z_b - t_a) \right) \prod_{1 \le a \le b \le k} \frac{1}{(t_b - t_a)(t_a - t_b + \hbar)} \right)$$

"weight functions"

$Gr_2\mathbb{C}^4$

Schur expansion

$$c^{Sm}(\Omega_{34}) = [\bar\Omega_{34}] + 3[\bar\Omega_{24}] + 4[\bar\Omega_{14}] + 4[\bar\Omega_{23}] +$$

$$+ 4[\bar\Omega_{13}] + [\bar\Omega_{12}]$$

(after putting $z_1 = z_2 = z_3 = z_4 = 0$
$\hbar = 1$)

$z_1 + z_2 - z_3 - z_4 + \underline{4\hbar}$

$(z_1 - z_3 + \underline{\hbar})(z_1 - z_4 + \underline{\hbar})(z_2 - z_3 + \underline{\hbar})(z_2 - z_4 + \underline{\hbar})$

$(z_1 - z_4 + \underline{2\hbar})(z_2 - z_4 + \underline{2\hbar})$

(sign behavior!)

$$\frac{c^{sm}\left(\Omega_{(3,1)}\right)}{c(TGr)} = s_{(3,1)} - \left(4\,s_{(4,1)} + 3\,s_{(3,2)} + 3\,s_{(3,1,1)}\right)$$

$$+ \left(10\,s_{(5,1)} + 13\,s_{(4,2)} + 5\,s_{(3,3)} + 10\,s_{(3,2,1)} + \right.$$

$$\left. + 6\,s_{(3,1,1)} + 13\,s_{(4,1)}\right) - \left(\cdots \right) + \left(\cdots \right) -$$

(sign behaviour!)

In $H^*(Gr_3 \mathbb{C}^6)$

$$\ydiagram{2,1} \cdot \ydiagram{2,1} = \ydiagram{3,3} + 2\,\ydiagram{3,2,1} + \ydiagram{2,2,2} + 11\,\ydiagram{3,3,1} + 11\,\ydiagram{3,2,2} + 46\,\ydiagram{3,3,2} + 108\,\ydiagram{3,3,3}$$

all $c^{sm}(\Omega_I)$ classes

(sign behavior!

Motivic Chern Classes

$$\left.\begin{array}{lll} [\bar\Omega_1] & z_2 - z_1 & 0 \\ [\bar\Omega_2] & 1 & 1 \end{array}\right)$$

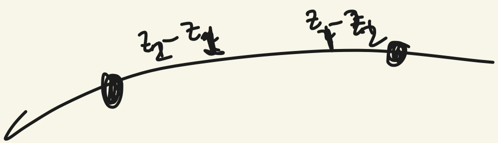
top $\hbar$-coeff.

$$\left.\begin{array}{lll} c^{sm}(\Omega_1) & z_2 - z_1 & 0 \\ c^{sm}(\Omega_2) & \hbar & z_1 - z_2 + \hbar \end{array}\right)$$

$$\begin{array}{lll} [\bar\Omega_1]^K & 1 - \frac{z_1}{z_2} & 0 \\ [\bar\Omega_2]^K & 1 & 1 \end{array}$$

$$\left.\begin{array}{lll} mC(\Omega_1) & 1 - \frac{z_1}{z_2} & 0 \\ mC(\Omega_2) & (1+\hbar)\frac{z_1}{z_2} & 1 + \frac{z_2 \hbar}{z_1} \end{array}\right\}$$

$$mC(\mathbb{P}^1) \quad 1 - \frac{z_1}{z_2} + (1+\hbar)\frac{z_1}{z_2} \qquad 1 + \frac{z_2}{z_1}\hbar$$



$$mC(\Omega_1) = \left(1 - \frac{z_1}{z_2},\ 0\right)$$

$$mC(\Omega_2) = \left((1+\hbar)\frac{z_1}{z_2},\ 1 + \hbar\frac{z_2}{z_1}\right)$$

$$\Sigma \quad 1 - \cancel{\frac{z_1}{z_2}} + \cancel{\frac{z_1}{z_2}} + \hbar\frac{z_1}{z_2}$$

$$c^{sm}(\Omega_I) \in H_T^*(Gr_k\mathbb{C}^n) \xrightarrow{Loc} \bigoplus_I \underbrace{H_T^*(x_I)}_{\mathbb{Z}[z_1, \dots z_n]}$$

$$\underbrace{mC}_{\text{"motivic Chern class"}}(\Omega_I) \in K_T(Gr_k\mathbb{C}^n) \xrightarrow{Loc} \bigoplus_I \underbrace{K_T(x_I)}_{\mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]}$$

im(Loc): (i-j)-neighboring components satisfy $z_i - z_j \mid f_I - f_J$

fact same description in K_T

$$z_i - z_j \mid f_I - f_J$$

$$\left(1 - \frac{z_j}{z_i} \mid f_I - f_J\right)$$

$H_T^*(\mathbb{P}^1)$ $\qquad\qquad$ $K_T(\mathbb{P}^1)$

$$[\bar{\Omega}_1] = (z_2 - z_1, \quad 0) \qquad\qquad [\bar{\Omega}_1]^K = \left(1 - \frac{z_1}{z_2}, \quad 0\right)$$

$$[\bar{\Omega}_2] = (1, \quad 1) \qquad\qquad [\bar{\Omega}_2]^K = (1, \quad 1)$$

$$c^{sm}(\Omega_1) = (z_2 - z_1, \quad 0) \qquad\qquad mC(\Omega_1) = \left(1 - \frac{z_1}{z_2}, \quad 0\right)$$

$$c^{sm}(\Omega_2) = (\hbar, \quad z_1 - z_2 + \hbar) \qquad\qquad mC(\Omega_2) = \left((1+\hbar)\frac{z_1}{z_2}, \quad 1 + \frac{z_2}{z_1}\hbar\right)$$

axiomatic definition

Thm-Def $\quad mC(\Omega_I) =$ unique class in $K_T(Gr_k \mathbb{C}^n)$

- $mC(\Omega_I)|_I = \displaystyle\prod_{\substack{i\in I\\ j\in \bar{I}\\ i<j}} \left(1 - \frac{z_i}{z_j}\right) \cdot \underbrace{\prod_{\substack{i\in I\\ j\in \bar{I}\\ i>j}} \left(1 + \hbar \frac{z_i}{z_j}\right)}_{c_I}$

- $mC(\Omega_I)|_J$ divisible by c_J

- $\underbrace{N}_{\text{Newton polygon}}\left(mC(\Omega_I)|_J\right) \subset \underbrace{N}_{\text{Newton polygon}}\left(mC(\Omega_J)|_J\right) - 0$ for $I \neq J$

(• $mC(\Omega_I)|_J = 0$ if $J \not\subset I$)

$$f \in \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}]$$

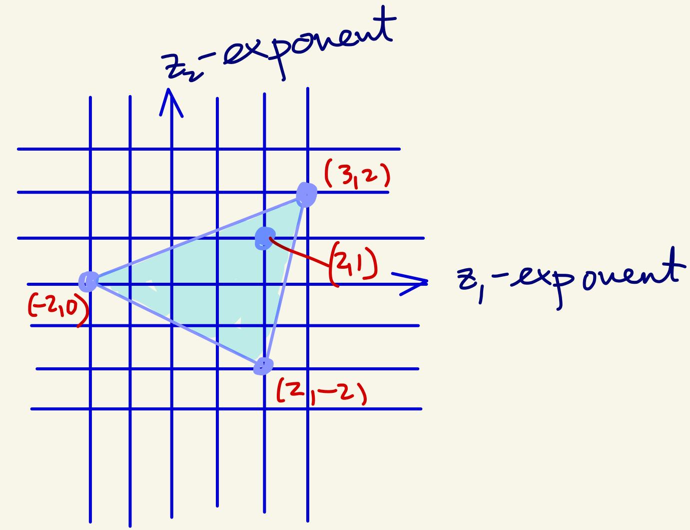
$$f = \sum_{K \in \mathbb{Z}^n} c_I \cdot \underline{z}^K$$

$$N(f) := \text{convex hull of } (K : c_K \neq 0)$$

ex

$$N\left(z_1^3 z_2^2 - 7\frac{z_1^2}{z_2^2} + 3 z_1^2 z_2 - 8\frac{1}{z_1^2} \right) =$$

$(3,2)$ $(2,-2)$ $(2,1)$ $(-2,0)$

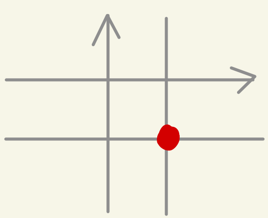


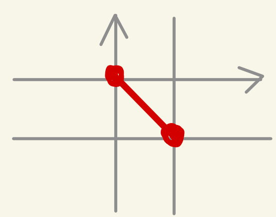
$$mC(\Omega_1) = \left(1 - \frac{z_1}{z_2},\ 0\right)$$

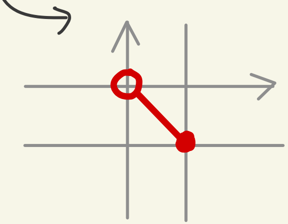
$$mC(\Omega_2) = \left((1+\hbar)\frac{z_1}{z_2},\ 1 + \frac{z_2}{z_1}\hbar\right)$$

Newton polygon axiom here:

$$N\left((1+\hbar)\frac{z_1}{z_2}\right) \subset N\left(1 - \frac{z_1}{z_2}\right) - 0$$







Where are we so far?

- axiomatic definition of $mC(\Omega_I) \in K_T(Gr)$

Facts

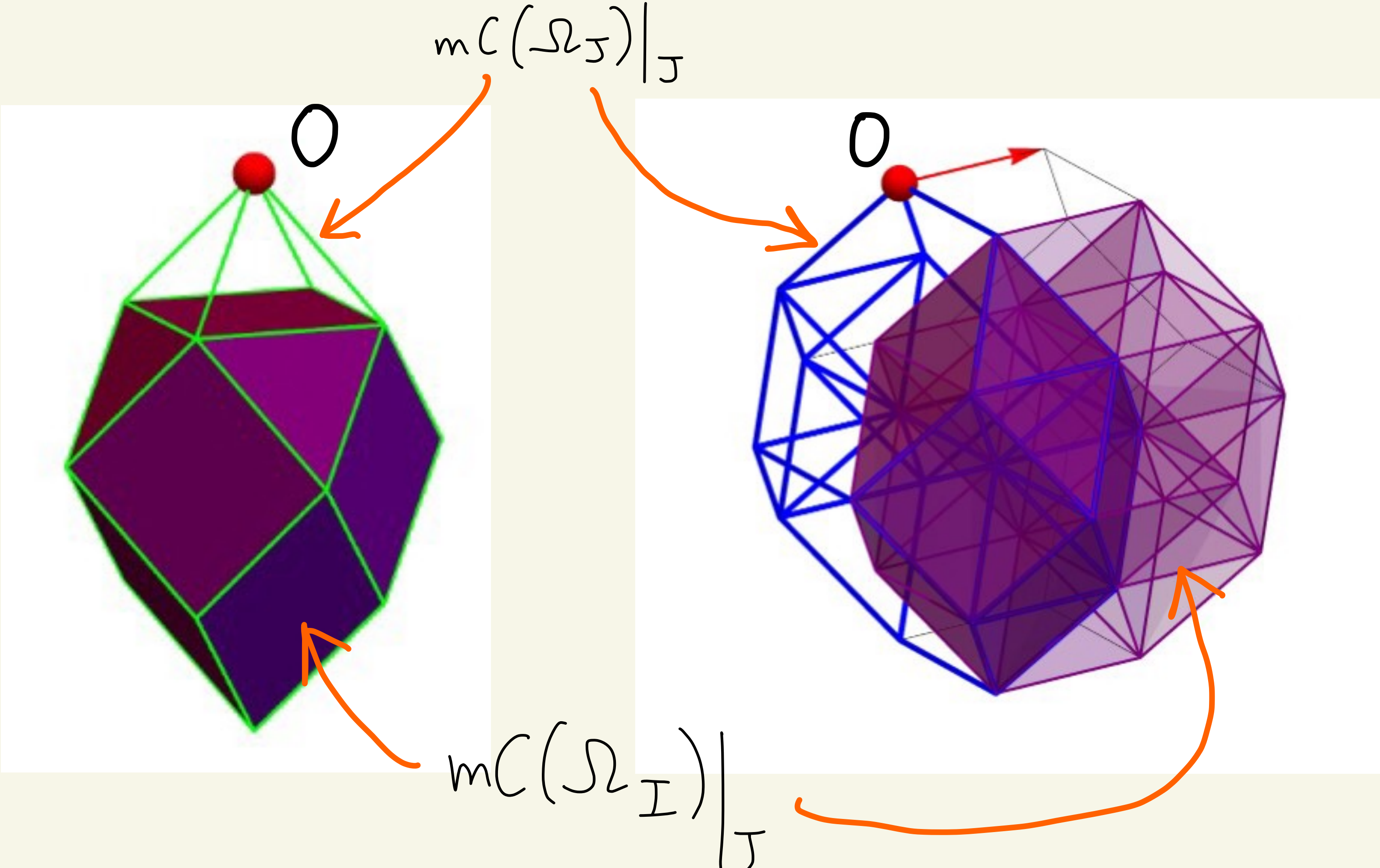
✓ • R-matrix property ("trigonometric solution of Yang-Baxter")

✓ • formulas of the type $\sum \prod \prod(\text{rational})$ exist ("trigonometric weight functions")

✓ • cotangent interpretation

✓ • Bott-Samelson & R-matrix recursions

✓ • MacPherson property ("motivic" class)


$mC(\Omega_J)|_J$
O
O
$mC(\Omega_I)|_J$

Enumerative geo of csm

$X \subset \mathbb{P}^N$ locally closed

- $X_r = X \cap \underbrace{H_1 \cap H_2 \cap \ldots \cap H_r}_{\text{general hyperplanes}}$

  $\chi_X(t) := \sum \chi(X_i)(-t)^i$

- $c^{sm}(X \subset \mathbb{P}^N) = \sum a_i \xi^i$

  $\gamma_X(t) := \sum a_i t^{N-i}$

$$\chi_X(t) \underset{\mathcal{J}}{\longleftrightarrow} \gamma_X(t)$$

involution on poly's

$$p \overset{\mathcal{J}}{\longmapsto} \frac{t \cdot p(-t-1) + p(0)}{t+1}$$