LINEAR STRUCTURES OF HODGE THEORY

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ABSTRACT. An introduction to Hodge structures and description of the linear structures underlying the Hard Lefschetz Theorem and Hodge–Riemann Bilinear Relations. Many exercises are given, those labelled with a $\star$ are the most important for this lecture.

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1. INTRODUCTION: DEFINITIONS, EXAMPLES AND GOAL FOR THIS LECTURE

Fix a field $\mathbb{F} \subset \mathbb{R}$. Let H be a finite dimensional vector space over $\mathbb{F}$.

1.1. Hodge structures.

Definition 1.1. A *Hodge structure* on H is given by a *Hodge decomposition* of the complexification $H_{\mathbb{C}} = H \otimes_{\mathbb{F}} \mathbb{C}$. The latter is a direct sum

$$H_{\mathbb{C}} \;=\; \bigoplus_{p,q\in\mathbb{Z}} H^{p,q} \tag{1.2a}$$

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with the property

$$\overline{H^{p,q}} \;=\; H^{q,p}\,. \tag{1.2b}$$

Example 1.3 (†). Historically Hodge structures first arose in the study of compact Kähler manifolds. Given one such manifold X, the Hodge Theorem asserts that the cohomology ring $H^\bullet(X,\mathbb{F})$ of admits a Hodge decomposition with $H^{p,q}(X) \subset H^\bullet(X,\mathbb{C})$ the cohomology classes represented by harmonic (p,q)–forms.

Familiarity with these geometric structures is not prerequisite for reading these notes – here we focus on *abstract Hodge structures*. This means that we will be concerned with relatively simple linear structures, not Kähler geometry (or any geometry). However, the historical significance of $H^\bullet(X,\mathbb{F})$ will merit the occasional comment – reader may regard this material – labeled with a † – as optional. The reader interested in Kähler geometry will find a brief summary of the Kähler package in §4.5. Finally, Christian Schnell's *Introduction to Hodge theory* is a highly recommended, elegant overview of the subject; this elementary talk can be found on the Simon's Center for Geometry and Physics "Video Portal".

Example 1.4. In the case that X is a toric variety, the $H^{p,q}(X)$ are nonzero only when $p = q$; that is, $H^\bullet(X,\mathbb{C}) = \oplus_k H^{k,k}(X)$, [CLS11].

Example 1.5 (†). Fix a lattice $\mathbb{Z}^{2g} \subset \mathbb{C}^g$, and consider the compact complex torus $X = \mathbb{C}^g/\mathbb{Z}^{2g}$. Given linear coordinates $(z_1,\ldots,z_g)$ on $\mathbb{C}^g$, the differentials $\{\mathrm{d}z_1,\ldots,\mathrm{d}z_g\}$ descend to well-defined closed 1-forms on X. The set of cohomology classes represented by the (p,q)–forms

$$\{\mathrm{d}z_{a_1}\wedge\cdots\wedge\mathrm{d}z_{a_p}\,\wedge\,\mathrm{d}\bar z_{b_1}\wedge\cdots\mathrm{d}\bar z_{b_q} \mid a_1<\cdots<a_p\,,\ b_1<\cdots<b_q\}$$

is a basis of $H^{p,q}(X)$.

Definition 1.6. The Hodge structure is *effective* if $H^{p,q} \neq \{0\}$ implies that $p,q \geq 0$.

Example 1.7. The Hodge structure of Example 1.3 is effective.

For convenience we will assume that

all Hodge structures discussed here are effective.

1.2. Pure Hodge structures.

Definition 1.8. The Hodge decomposition (1.2) defines a *pure Hodge structure of weight* $\mathsf{w} \in \mathbb{Z}$ if $H^{p,q} = 0$ for all $p+q \neq \mathsf{w}$. In this case

$$H_{\mathbb{C}} \;=\; \bigoplus_{p+q=\mathsf{w}} H^{p,q}\,. \tag{1.9}$$

Example 1.10. Every vector space H trivially admits a pure Hodge structure of weight $\mathsf{w} = 2k$ given by $H_{\mathbb{C}} = H^{k,k}$.

Example 1.11 (†). The Hodge Theorem asserts that the w–th cohomology group $H^{\mathsf{w}}(X, \mathbb{F})$ admits a pure Hodge structure of weight w with Hodge summands $H^{p,q}(X)$, $p+q=\mathsf{w}$.

Remark 1.12. A pure, effective Hodge structure necessarily has non-negative weight $\mathsf{w} \geq 0$.

Definition 1.13. A pure *Hodge structure* of weight $\mathsf{w} \geq 0$ on H is given by a *Hodge filtration* of the complexification $H_{\mathbb{C}} = H \otimes_{\mathbb{F}} \mathbb{C}$. The latter is a (finite) decreasing filtration

$$0 \subset F^{\mathsf{w}} \subset F^{\mathsf{w}-1} \subset \cdots \subset F^1 \subset F^0 = H_{\mathbb{C}} \tag{1.14a}$$

satisfying

$$H_{\mathbb{C}} = F^k \oplus \overline{F^{\mathsf{w}+1-k}} . \tag{1.14b}$$

Exercise 1.15. The definitions 1.8 and 1.13 are equivalent:

(a) Given a pure Hodge decomposition (1.9), show that $F^k = \oplus_{p \geq k} H^{p,q}$ defines a Hodge filtration (1.14).

(b) Given a Hodge filtration (1.14), show that $H^{p,q} = F^p \cap \overline{F^q}$ defines a pure Hodge decomposition (1.9).

1.3. **Polarized Hodge structures.**

Definition 1.16. A *polarization* of a pure Hodge structure of weight w is given by a nondegenerate bilinear form $Q : H \times H \to \mathbb{F}$ satisfying

$$Q(v,u) = (-1)^{\mathsf{w}} Q(u,v) , \quad \forall\, u, v \in H ,$$

and the *Hodge–Riemann bilinear relations*

$$Q(F^k, F^{\mathsf{w}+1-k}) = 0 , \tag{1.17a}$$

$$\mathbf{i}^{p-q} Q(u, \bar{u}) > 0 , \quad \forall\, 0 \neq u \in H^{p,q} . \tag{1.17b}$$

Example 1.18. Recall the trivial, pure Hodge structure $H_{\mathbb{C}} = H^{k,k}$ of weight $\mathsf{w} = 2k$ (Example 1.10). A polarization of this Hodge structure is nothing more than an inner-product on H.

Example 1.19 (†). Let X be a projective Kahler manifold of dimension n with Kähler class $\omega \in H^{1,1} \cap H^2(X, \mathbb{R})$. Given $\mathsf{w} \leq n$, the *primitive cohomology*

$$P_{\mathsf{w}} := \{ \alpha \in H^{\mathsf{w}}(X, \mathbb{R}) \mid \omega^{n-\mathsf{w}+1} \wedge \alpha = 0 \}$$

inherits a weight w Hodge decomposition $P_{\mathsf{w},\mathbb{C}} = \oplus_{p+q=\mathsf{w}} P^{p,q}_{\mathsf{w}}$ from $H^{\mathsf{w}}(X, \mathbb{R})$ given by

$$P^{p,q}_{\mathsf{w}} = H^{p,q}(X) \cap P_{\mathsf{w},\mathbb{C}} .$$

This Hodge structure is polarized by

$$Q(\alpha, \beta) = (-1)^{\mathsf{w}(\mathsf{w}-1)} \int_X \alpha \wedge \beta \wedge \omega^{n-\mathsf{w}} .$$

1.4. **Goal of this lecture.** The Hodge–Riemann bilinear relations of Example 1.19 are consequences of a nice linear structure on the real vector space $H^\bullet(X,\mathbb{R})$:

(i) The Lefschetz operator $M : H^\bullet(X,\mathbb{R}) \to H^\bullet(X,\mathbb{R})$ mapping $\alpha \mapsto \omega\wedge\alpha$ can be completed to an $\mathfrak{sl}(2,\mathbb{R})$–triple that is compatible with the inner product and the Hodge decomposition $H^\bullet(X,\mathbb{C}) = \oplus\, H^{p,q}(X,\mathbb{C})$.

(ii) A Hermitian structure on $H^\bullet(X,\mathbb{R})$ that is compatible (in a sense to be made precise, §§4.3–4.4) with the $\mathfrak{sl}(2,\mathbb{R})$ action.

The goal of this lecture is to explain how such linear structures yield the Hard Lefschetz Theorem and the Hodge–Riemann Bilinear Relations. These notes include a number of exercises; the "key exercises" as marked with a $\star$.

For a thorough treatment of Hodge theory see [CMSP17].

2. Representations of $\mathfrak{sl}(2)$: Lefschetz

The punchline of this section is Exercises 2.20 and 2.21, describing the linear structure underpinning the Hard Lefschetz Theorem. The material covered here is classical; there are many excellent references, including the accessible [EW06, §8].

Let $\mathfrak{sl}(2,\mathbb{F})$ denote the vector space of 2×2, trace-free matrices with entries in $\mathbb{F}$. A basis is given by

$$\mathbf{m} \;=\; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{y} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{n} \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The following exercise asserts that $\mathfrak{sl}(2,\mathbb{F})$ is a Lie algebra: it is closed under the commutator Lie bracket.

Exercise 2.1. (a) Given $A,B\in\mathfrak{sl}(2,\mathbb{F})$, the commutator

$$[A,B] \;=\; AB \,-\, BA$$

is also an element of $\mathfrak{sl}(2,\mathbb{F})$.

(b) Show that

$$[\mathbf{y},\mathbf{m}] \;=\; 2\,\mathbf{m}\,, \quad [\mathbf{m},\mathbf{n}] \;=\; \mathbf{y}\,, \quad [\mathbf{n},\mathbf{y}] \;=\; 2\,\mathbf{n}\,.$$

Let V be a $\mathbb{F}$–vector space. Let $\mathrm{End}(V)$ be the $\mathbb{F}$–vector space of all linear maps $V\to V$.

Exercise 2.2. Show that $\mathrm{End}(V)$ is isomorphic to the vector space of $m\times m$ matrices with entries in $\mathbb{F}$, with $m=\dim V$. [*Hint.* Fix a basis $\{e_i\}$ of V, and consider the action of $X\in\mathrm{End}(V)$ on e_i.]

Given $A,B\in\mathrm{End}(V)$ the commutator $[A,B] = A\circ B - B\circ A$ is also a linear map $V\to V$. So $\mathrm{End}(V)$ also has the structure of a Lie algebra; this is the *endomorphism algebra.*

Definition 2.3. A *representation of* $\mathfrak{sl}(2,\mathbb{F})$ is given by a $\mathbb{F}$–vector space V, and a triple $\{M,Y,N\} \subset \mathrm{End}(V)$ satisfying

$$[Y,M] \;=\; 2\,M\,, \quad [M,N] \;=\; Y\,, \quad [N,Y] \;=\; 2\,N\,.$$

Remark 2.4. In this case $\mathfrak{g} = \mathrm{span}\{M,Y,N\}$ is a Lie subalgebra of $\mathrm{End}(V)$ that is isomorphic to $\mathfrak{sl}(2,\mathbb{F})$, and the linear map $\mathfrak{sl}(2,\mathbb{F}) \to \mathrm{End}(V)$ defined by $\mathfrak{m} \mapsto M$, $\mathbf{y} \mapsto Y$ and $\mathbf{n} \mapsto N$ is an injective Lie algebra homomorphism.

Exercise 2.5 $(\star)$. Let

$$S_d \;=\; \mathrm{span}_{\mathbb{F}}\{x^a y^b \mid a+b=d\} \;\subset\; \mathbb{F}[x,y]$$

be the vector space of degree d homogeneous polynomials in two variables.

(a) Prove that
$$M \;=\; x\frac{\partial}{\partial y}\,, \quad Y \;=\; x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,, \quad N \;=\; y\frac{\partial}{\partial x}$$
is an $\mathfrak{sl}(2,\mathbb{F})$–triple.

(b) Prove that $x^a\,y^b$ is an eigenvector of Y with eigenvalue $a-b \in \mathbb{Z}$. In particular, the Y–eigenvalues
$$\{d, d-2, d-4, \ldots, 4-d, 2-d, -d\}$$
are symmetric about the origin.

(c) Prove that $\{y^d\,,\, M(y^d)\,,\, M^2(y^d)\,, \cdots, M^d(y^d)\}$ is a basis of S_d.

(d) Prove that $M^{d+1}=0$ and $N^{d+1}=0$.

The picture is

$$0 \xleftarrow{\;M\;} \mathbb{C}x^d \underset{M}{\overset{N}{\rightleftarrows}} \mathbb{C}x^{d-1}y \overset{N}{\rightarrow} \cdots \overset{}{\underset{M}{\leftarrow}} \mathbb{C}xy^{d-1} \underset{M}{\overset{N}{\rightleftarrows}} \mathbb{C}y^d \xrightarrow{\;N\;} 0\,.$$

Remark 2.6. Assume that $\mathbb{F} = \overline{\mathbb{F}}$ is algebraically closed. The content of Theorem 2.11 is that the S_d are the building blocks of $\mathfrak{sl}(2,\mathbb{F})$ representations: every representation can be decomposed into a direct sum of the S_d, and the S_d are themselves "irreducible".

Definition 2.7. A *subrepresentation* is a linear subspace $U \subset V$ that is invariant under the action of $\mathfrak{sl}(2,\mathbb{F})$; that is, $M(U), Y(U), N(U) \subset U$. The representation V of $\mathfrak{sl}(2,\mathbb{F})$ is *irreducible* if there exists no nontrivial subspace $0 \subsetneq U \subsetneq V$; that is, $M(U), Y(U), N(U) \subset U$ implies $U=0$ or $U=V$.

Exercise 2.8 $(\star)$. The representation S_d is irreducible.

Definition 2.9. The vector space S_d is the irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$ of *highest weight d*.

Definition 2.10. Two $\mathfrak{sl}(2,\mathbb{F})$–representations V and U are *isomorphic*, and we write $V \simeq U$, if there is a linear isomorphism $\lambda : V \to U$ such that $\lambda(X(v)) = X(\lambda(v))$ for all $v \in V$ and $X \in \mathfrak{sl}(2,\mathbb{F})$.

For the remainder of §2 we

assume $\mathbb{F} = \overline{\mathbb{F}}$ is algebraically closed.

Theorem 2.11 (Weyl)**.** *Every finite–dimensional representation V of $\mathfrak{sl}(2,\mathbb{F})$ is completely reducible: there exist $0 \le m_d \in \mathbb{Z}$ so that*

$$V \ \simeq \ \bigoplus_{d\ge 0} S_d^{\oplus m_d} \tag{2.12}$$

as $\mathfrak{sl}(2,\mathbb{F})$–representations.

Definition 2.13. We call the summand $I_d \simeq S_d^{m_d}$ in (2.12) the *isotypic component of highest weight d*, and m_d is the *multiplicity* of S_d in V.

Exercise 2.14. (a) Suppose that V is an $\mathfrak{sl}(2,\mathbb{F})$–representation, and that the eigenvalues of Y are $\{2,1,1,0,-1,-1,-2\}$ (listed with multiplicity). Prove that the decomposition (2.12) is $V = S_2 \oplus S_1^{\oplus 2}$.

(b) Prove that the decomposition (2.12) is determined by the eigenvalues of Y; that is, show that Y–eigenvalues, listed with multiplicity, determine m_d.

Given $k \in \mathbb{Z}$, define

$$V_k \ = \ \{v \in V \mid Y(v) = k\}\,. \tag{2.15}$$

Example 2.16. If $V = S_d$, then $\mathbb{C}x^a x^b = V_{a-b}$ (with $a+b=d$, Exercise 2.5).

Exercise 2.17 ($\star$). (a) Show that $\dim V_k = \dim V_{-k}$.

(b) Show that $\dim V_{k+2} \le \dim V_k$ for all $k \ge 0$. That is, the two sequences

$$\{\ldots,\, \dim V_4\,,\, \dim V_2\,,\, \dim V_0\,,\, \dim V_2\,,\, \dim V_{-4}\,,\ldots\}$$
$$\{\ldots,\, \dim V_3\,,\, \dim V_1\,,\, \dim V_{-1}\,,\, \dim V_{-3}\,,\ldots\}$$

are *unimodal*.

Exercise 2.18. Prove that

$$V_{\mathrm{even}} \ = \ \bigoplus_k V_{2k} \quad \text{and} \quad V_{\mathrm{odd}} \ = \ \bigoplus_k V_{2k+1} \tag{2.19}$$

are both subrepresentations of V (Definition 2.7).

Exercise 2.20 ($\star$). Suppose that $k \ge 0$ and prove that $M^k : V_{-k} \to V_k$ is an isomorphism.

Exercise 2.21 ($\star$). Let V be a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{F})$. Given $d \geq 0$, define

$$P_d \;=\; \{v \in V \mid Y(v) = -d\,,\ M^{d+1}(v) = 0\}\,. \tag{2.22}$$

(a) Prove that $\dim P_d = m_d$.

(b) Prove that the isotypic component of highest weight d is

$$I_d \;=\; \bigoplus_{k=0}^{d} M^k(P_d)$$

(c) Conclude that

$$V \;=\; \bigoplus_{d\geq 0} \bigoplus_{k=0}^{d} M^k(P_d)\,.$$

Exercise 2.23. Show that $\oplus P_d = \ker\{N : V \to V\}$.

3. Representations of $\mathfrak{sl}(2)$: compatible bilinear forms

Lurking in the background of the Hodge–Riemannian bilinear relations are some basic results (Exercises 3.3–3.4) about $\mathfrak{sl}(2,\mathbb{R})$–representations that are compatible with a bilinear form. These are implicit in the constructions of §4, and the discussion here is not necessary for that material. But while the construction of §4 has the advantage of being explicit, it has the disadvantage of being somewhat complicated and so obscuring those aforementioned basic results. So here we outline the essential underlying linear structure.

Let $\mathrm{Aut}(V)$ be the group of invertible linear maps $V \to V$.

Exercise 3.1. Show that $\mathrm{Aut}(V)$ is isomorphic to the group $\mathrm{GL}_m\mathbb{F}$ of invertible $m \times m$ matrices with entries in $\mathbb{F}$. [*Hint.* See Exercise 2.2.]

Fix $\mathbb{F} = \mathbb{R}$, and let

$$Q : V \times V \;\to \mathbb{R}$$

be a nondegenerate (skew-)symmetric bilinear form. Then the group of automorphisms

$$\mathrm{Aut}(V,Q) \;=\; \{A \in \mathrm{Aut}(V) \mid Q(Au,Av) = Q(u,v)\,,\ \forall u,v \in V\}$$

is $\mathrm{Sp}(2r,\mathbb{R})$ if Q is skew-symmetric, and $\mathrm{O}(a,b)$ if Q is symmetric. Likewise the associated Lie algebra

$$\mathrm{End}(V,Q) \;=\; \{X \in \mathrm{End}(V) \mid Q(Xu,v) + Q(u,Xv) = 0\,,\ \forall u,v \in V\}$$

is either $\mathfrak{sp}(2r,\mathbb{R})$ or $\mathfrak{o}(a,b)$.

Exercise 3.2. Prove that $\mathrm{End}(V,Q)$ is a Lie subalgebra of $\mathrm{End}(V)$; that is, if $A,B \in \mathrm{End}(V,Q)$, then $[A,B] = A\circ B - B\circ A \in \mathrm{End}(V,Q)$.

An $\mathfrak{sl}(2,\mathbb{R})$–representation is *compatible* with Q if

$$\{M, Y, N\} \ \subset \ \mathrm{End}(V, Q)\,.$$

Exercise 3.3 ($\star$). Given a Q–compatible $\mathfrak{sl}(2,\mathbb{R})$–representation, show that:

(a) If $k + \ell \neq 0$, then $Q(V_k, V_\ell) = 0$; the pairing $Q : V_k \times V_{-k} \to \mathbb{R}$ is perfect.

(b) If $c \neq d$, then $Q(I_c, I_d) = 0$.
In particular, the isotypic decomposition $V = \oplus_d I_d$ is Q–orthogonal.

Specify $s \in \{0, 1\}$ so that

$$Q(v, u) \ = \ (-1)^s \, Q(u, v)\,.$$

(That is, $s = 0$ if Q is symmetric, and $s = 1$ if Q is skew-symmetric.) Define a bilinear form $Q_d : P_d \times P_d \to \mathbb{R}$ by

$$Q_d(u, v) \ = \ Q(u, M^d v)\,.$$

Exercise 3.4 ($\star$). (a) Prove that Q_d is nondegenerate. (This result, combined with Exercise 3.3(b), gives us a weak form of the Hodge–Riemann bilinear relations.)

(b) Prove that $Q_d(v, u) = (-1)^{s+d} Q_d(u, v)$.

Example 3.5 ($\star$). We can define a pure Hodge structure of weight $\mathsf{w} = 2k$ on P_d by specifying that $P_{d,\mathbb{C}} = P^{k,k}$, cf. Example 1.10. If $s + d \equiv 0 \bmod 2$, so that Q_d is symmetric, then the nondegenerate Q_d is a polarization if and only if it is positive definite, cf. Example 1.18.

4. Linear structures underlying the "Kähler package"

We now turn to the linear structures underlying the Hodge–Riemann bilinear relations. One begins with a Hermitian structure on a real vector space T that is given by an inner product $\langle \cdot, \cdot \rangle$ and a compatible complex structure $J : T \to T$. From this data one constructs an $\mathfrak{sl}(2)$–triple acting on the exterior algebra $V = \bigwedge^\bullet T$, cf. §A. The complex structure naturally endows the exterior algebra with a Hodge structure, which the primitive subspace (2.22) inherits. The inner-product and $\mathfrak{sl}(2)$ action (in particular, the Lefschetz operator) endow the primitive subspace with a polarization satisfying the Hodge–Riemann bilinear relations.

The material here largely follows the discussion of [Huy05, §1.2].

4.1. **Hermitian structures.**
Let T be a real, finite dimensional vector space.

Definition 4.1. A linear map $J : T \to T$ is a *complex structure* if $J^2 = -\mathrm{id}$.

Exercise 4.2. (a) Show that the complexification $T_\mathbb{C} = T \otimes_\mathbb{R} \mathbb{C}$ decomposes as a direct sum $T^{1,0} \oplus T^{0,1}$ of J–eigenspaces, with eigenvalues $\pm\mathbf{i}$, respectively.

(b) Prove that $\overline{T^{1,0}} = T^{0,1}$.

(c) Prove that T admits a basis $\{x_1, y_1, \ldots, x_n, y_n\}$ so that $J(x_a) = -y_a$.

(d) Prove that $\{z_a = x_a + \mathbf{i} y_a\}_{a=1}^n$ is a basis of $T^{1,0}$.

(e) Define an orientation on T by specifying that $\{x_1, y_1, \ldots, x_n, y_n\}$ is an oriented basis. Prove that this definition is independent of our choice of basis.

Exercise 4.3. Prove that the k–th exterior exterior power (§A) decomposes as

$$\bigwedge^k T_{\mathbb{C}} \;=\; \bigoplus_{p+q=k} \bigwedge^{p,q} T\,, \qquad \text{where } \bigwedge^{p,q} T \;=\; (\bigwedge^p T^{1,0}) \otimes (\bigwedge^q T^{0,1})\,. \tag{4.4}$$

Note that $\overline{\bigwedge^{p,q} T} = \bigwedge^{p,q} T$ and (4.4) is a pure Hodge decomposition of weight k.

Fix an inner-product $\langle \cdot, \cdot \rangle$ on T that is compatible with the complex structure; that is,

$$\langle u, v \rangle \;=\; \langle J(u), J(v) \rangle\,, \qquad \forall\; u, v \in T\,.$$

Given an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of T, we specify an inner-product on $\bigwedge^k T$ by declaring $\{e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le 2n\}$ to be an orthonormal basis.

Exercise 4.5. Show that you can choose the basis in Exercise 4.2(c) so that $\{x_a, y_a\}_{a=1}^n$ is also orthonormal.

4.2. **Lefschetz operator.**

Definition 4.6. The *fundamental form* is

$$\omega \;=\; \tfrac{\mathbf{i}}{2} \sum_a z_a \wedge \overline{z}_a \;=\; \sum_a x_a \wedge y_a \;\in\; (\bigwedge^2 T) \cap (\bigwedge^{1,1} T)\,.$$

The inner product on T naturally induces one on T^*. If $\{e_1, \ldots, e_{2m}\}$ is an orthonormal basis of T, then we declare the dual basis of T^* to be orthonormal as well. This inner product is also denoted $\langle \cdot, \cdot \rangle$.

Exercise 4.7. Show that $\omega(u, v) = \langle J(u), v \rangle = -\langle u, J(v) \rangle$ for all $u, v \in T^*$. In particular, ω is independent of our choice of basis.

Definition 4.8. The *Lefschetz operator*

$$M : \bigwedge^{\bullet} T \;\to\; \bigwedge^{\bullet} T$$

maps $\alpha \mapsto \omega \wedge \alpha$.

Observe that $M(\bigwedge^k T) \subset \bigwedge^{k+2} T$. And, extending M to a $\mathbb{C}$–linear operator $\bigwedge^{\bullet} T_{\mathbb{C}} \to \bigwedge^{\bullet} T_{\mathbb{C}}$, we have

$$M(\bigwedge^{a,b} T) \;\subset\; \bigwedge^{a+1,b+1} T\,. \tag{4.9}$$

Definition 4.10. The *dual Lefschetz operator*

$$N : \textstyle\bigwedge^\bullet T \ \to \ \bigwedge^\bullet T$$

is the adjoint to M; that is

$$\langle N\alpha, \beta\rangle \ = \ \langle \alpha, M\beta\rangle \,.$$

The dual Lefschetz operator may be expressed as

$$N \ = \ *^{-1} \circ M \circ * \,, \tag{4.11}$$

where $*$ is the *Hodge $*$–operator*. The latter is defined as follows. Set

$$vol \ = \ (x_1 \wedge y_1) \wedge \cdots \wedge (x_n \wedge y_n) \ \in \ \textstyle\bigwedge^{2n} T \,.$$

Define

$$* : \textstyle\bigwedge^k T \ \to \bigwedge^{2n-k} T$$

by

$$\alpha \wedge *\beta \ = \ \langle \alpha, \beta\rangle vol \,.$$

Exercise 4.12. (a) Given an oriented, orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of T and a disjoint union $I \cup J = \{1, \ldots, 2m\}$ we have

$$*e_{i_1} \wedge \cdots \wedge e_{i_k} \ = \ \pm e_{j_1} \wedge \cdots \wedge e_{j_{2n-k}} \,,$$

where $\pm$ is the sign of the permutation $(I, J) = (i_1, \ldots, i_k\,, j_1, \ldots, j_{2n-k}) \in \mathfrak{S}_{2n}$, cf. Definition A.4.

(b) Prove $*1 = vol$.

(c) Given $\alpha \in \bigwedge^k T$, show that $*^2\alpha = (-1)^{k(2n-k)}\alpha$.

(d) Given $\alpha \in \bigwedge^k T$, show that $\langle \alpha, *\beta\rangle = (-1)^{k(2n-k)}\langle *\alpha, \beta\rangle$. (This says the Hodge $*$ operator is self-adjoint up to a sign.)

Exercise 4.13 ($\star$). (a) Verify (4.11).

(b) Show that $N(\bigwedge^{p,q} T) \subset \bigwedge^{p-1,q-1} T$.

4.3. **The $\mathfrak{sl}(2)$–triple.** Define

$$Y : \textstyle\bigwedge^\bullet T \ \to \ \bigwedge^\bullet T$$

by specifying that Y acts on $\bigwedge^k T$ by $(k-n)$id; that is, $\bigwedge^\bullet T = \oplus_k \bigwedge^k T$ is the Y–eigenspace decomposition, and the Y–eigenvalue of $\bigwedge^k T$ is $k - n$.

Exercise 4.14 ($\star$). Prove that

$$[Y, M] \ = \ 2M \quad \text{and} \quad [N, Y] \ = \ 2N \,. \tag{4.15}$$

To see that $\{M, Y, N\}$ is an $\mathfrak{sl}(2)$–triple (§2) it remains to show that

$$[M, N] \;=\; Y\,. \tag{4.16}$$

The trick is to reduce the proof of (4.16) to the case that $\dim T = 2$. For this, the basic idea is that if $T = T_1 \oplus T_2$ is orthogonal and preserved by the complex structure, then $\omega = \omega_1 + \omega_2$ with ω_i the fundamental form of T_i. Since $\bigwedge^\bullet T = (\bigwedge^\bullet T_1) \otimes (\bigwedge^\bullet T_2)$ it follows that $M = M_1 \otimes \mathrm{id} + \mathrm{id} \otimes M_2$. See [Huy05, Prop. 1.2.26] for details.

Exercise 4.17. Verify (4.16) for $\dim T = 2$.

Remark 4.18. It follows from (4.15) and (4.16) that $\{M, Y, N\}$ is an $\mathfrak{sl}(2)$–triple; in particular,

$$\boxed{V = \textstyle\bigwedge^\bullet T \text{ is an } \mathfrak{sl}(2, \mathbb{R})\text{–representation.}}$$

While Theorem 2.11 and Exercises 2.20–2.21 are stated for an algebraically closed field; the same results hold here for the real representation. This is because:

- The complex structure $J : T \to T$ gives T the structure of a complex vector space; scalar multiplication by $a + \mathbf{i}b \in \mathbb{C}$ is defined to be $(a + \mathbf{i}b)v = av + bJ(v)$ for all $v \in T$.
- The action of the triple $\{M, Y, N\}$ on V commutes with the complex structure J; in particular the action is complex linear.

Remark 4.19. Recall the Y–eigenspace V_k of (2.15). Observe that $V_k = \bigwedge^{n+k} T$; equivalently,

$$\textstyle\bigwedge^\ell T \;=\; V_{\ell - n}$$

Exercise 4.20. Let I_d be the isotypic component of highest weight d in $\bigwedge^\bullet T$, cf. Definition 2.13. Prove that $I_d = 0$ for all $d > n$. Equivalently, $m_d = 0$ in (2.12) for all $d > n$.

Define

$$\textstyle\bigwedge^{\mathrm{even}} T \;=\; \oplus \bigwedge^{2k} T \quad \text{and} \quad \bigwedge^{\mathrm{odd}} T \;=\; \oplus \bigwedge^{2k+1} T\,.$$

Exercise 4.21. Prove that $\bigwedge^{\mathrm{even}} T$ and $\bigwedge^{\mathrm{odd}} T$ are both subrepresentations of $\bigwedge^\bullet T$, cf. Definition 2.7 and Exercise 2.18.

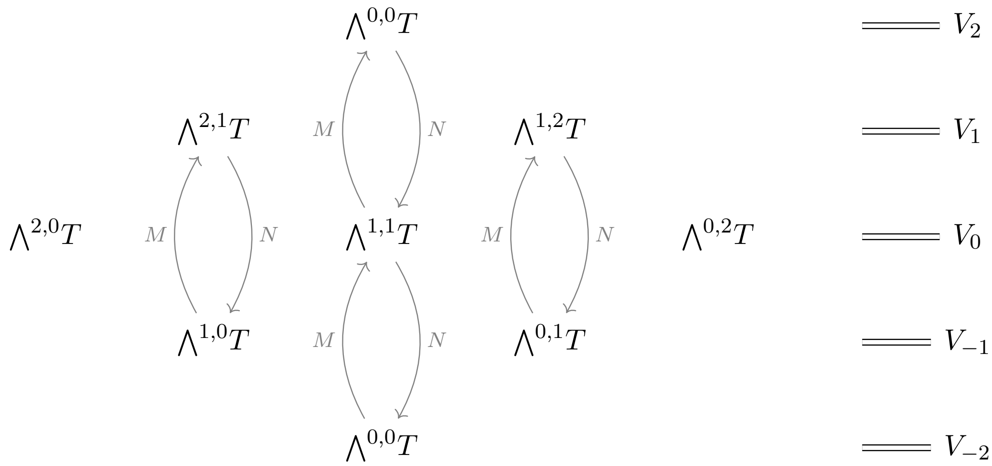
Exercise 4.22. (a) Fix $0 \le k \le n$. Prove that

$$U \;=\; \textstyle\bigwedge^{n,k} T \;\oplus\; \bigwedge^{n-1,k-1} T \;\oplus\; \bigwedge^{n-2,k-2} T \;\oplus \cdots \oplus\; \bigwedge^{n-k,0}$$

is a subrepresentation $\bigwedge^\bullet T_{\mathbb{C}}$.

(b) Let I_d be the isotypic component of highest weight d in U, cf. Definition 2.13. Prove that $I_d = 0$ for all $d > k$; equivalently, $m_d = 0$ for all $d > k$.

Example 4.23. In the case that $n=2$ we may visualize the representation on $\bigwedge^\bullet T$ as



Note that the vertical subrepresentations are always center about the middle row V_0 (Exercise 2.5(b)).

Exercise 4.24 ($\star$). Recall the primitive space (2.22).

(a) Show that $P_d \subset \bigwedge^{n-d}T$.

(b) Given $p+q=n-d$, define $P^{p,q} = P_{d,\mathbb{C}} \cap \bigwedge^{p,q}T$. Use (4.9) to show that

$$P_{d,\mathbb{C}} \;=\; \bigoplus_{p+q=n-d} P^{p,q}\,. \tag{4.25}$$

(c) Prove that (4.25) is a pure Hodge decomposition of weight $\mathsf{w} = n-d$.

(d) Show that

$$\bigwedge^{p,q}T \;=\; \bigoplus M^k(P^{p-k,q-k})\,.$$

4.4. **The Hodge–Riemann bilinear relations.** Extend the inner-product $\langle\cdot,\cdot\rangle$ on $V = \bigwedge^\bullet T$ to a Hermitian form on the complexification $V_\mathbb{C} = \bigwedge^\bullet T_\mathbb{C}$ by specifying

$$\langle \lambda u, \mu v\rangle \;=\; \lambda\bar\mu\langle u,v\rangle\,,$$

for all $\lambda,\mu\in\mathbb{C}$ and $u,v\in\bigwedge^\bullet T$.

Example 4.26. Recall the basis $\{z_a\}_{a=1}^n \subset T^{1,0}$ of Exercise 4.2. We have $\langle z_a, z_b\rangle = \langle \overline{z}_a, \overline{z}_b\rangle = 2\delta_{\alpha\beta}$ and $\langle z_a, \overline{z}_b\rangle = 0$.

Extend the Hodge $*$ operator to a $\mathbb{C}$–linear $\bigwedge^\bullet T_\mathbb{C} \to \bigwedge^\bullet T_\mathbb{C}$.

Exercise 4.27. (a) Given $\alpha,\beta\in\bigwedge^\bullet T_\mathbb{C}$, verify that $\alpha\wedge *\overline{\beta} = \langle\alpha,\beta\rangle vol$.

(b) Show that $*(\bigwedge^{p,q}T) \subset \bigwedge^{n-q,n-p}T$.

(c) Show that the Hodge decomposition $\bigwedge^\bullet T_{\mathbb{C}} = \oplus \bigwedge^{p,q} T$ is orthogonal with respect to the Hermitian pairing.

Definition 4.28. Given $k \le n$, the *Hodge–Riemann pairing*

$$Q_k : \bigwedge^k T \times \bigwedge^k T \to \mathbb{R}$$

is given by

$$Q_k(\alpha, \beta) vol = (-1)^{k(k-1)/2} \alpha \wedge \beta \wedge \omega^{n-k} .$$

We also let Q_k denote the extension to a $\mathbb{C}$–bilinear pairing on $\bigwedge^k T_{\mathbb{C}}$.

Exercise 4.29 ($\star$). (a) Show that $Q_k(\alpha, \beta) = (-1)^k Q_k(\beta, \alpha)$. [*Hint.* Exercise A.11.]

(b) Given $p + q = k = r + s$, show that $Q_k(\bigwedge^{p,q} T , \bigwedge^{r,s} T) = 0$ if $(r, s) \neq (q, p)$.

In order to show that Q_k polarizes the weight k Hodge structure on $P_{n-k} \subset \bigwedge^k T$ it remains to establish

$$\mathbf{i}^{p-q} Q_k(\alpha, \overline{\alpha}) = (n-k)! \langle \alpha, \alpha \rangle , \tag{4.30}$$

for all $\alpha \in P^{p,q}$. We will need the following fact

$$* M^{n-k} \alpha = (-1)^{k(k+1)/2} (n-k)! \, \mathbf{i}^{p-q} \alpha , \tag{4.31}$$

cf. [Huy05, Prop. 1.2.31]; the argument is again by induction on $\dim T$, but with $\dim_{\mathbb{R}} T_1 = 2$. Define $\beta \in \bigwedge^k T$ by $* \overline{\beta} = M^{n-k} \overline{\alpha}$. Then Exercise 4.12(c) and (4.31) yield

$$\beta = (-1)^{k + k(k+1)/2} (n-k)! \, \mathbf{i}^{p-q} \alpha .$$

The desired (4.30) follows from

$$\begin{aligned} Q_k(\alpha, \overline{\alpha}) &= (-1)^{k(k-1)/2} \alpha \wedge (M^{n-k} \overline{\alpha}) \\ &= (-1)^{k(k-1)/2} \langle \alpha, \beta \rangle vol . \end{aligned}$$

4.5. **The Kähler package** (†). For the reader interested in Kähler geometry, what one has in the back of one's mind is that $T = T_p^* X$ is the (real) cotangent space of a compact Kähler manifold X. Then $T^{1,0}$ is the holomorphic cotangent space at p, and $T^{0,1}$ is its conjugate. The inner product $\langle \cdot, \cdot \rangle$ is dual to the Riemannian metric on $T_p X$, and the fundamental form ω is the Kähler form at p. One obtains the Hard Lefschetz Theorem and Hodge–Riemann bilinear relations for the cohomology $H^\bullet(X)$ as follows:

(i) Let $\mathcal{A}^k(X)$ denote the space of smooth, complex-valued k forms; and the $\mathcal{A}^{p,q}(X)$ the space of smooth (p, q)–forms. The Kähler form defines the Lefschetz operator $M : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q+1}(X)$ point-wise on X. The Hodge $*$ operator $\mathcal{A}^k(X) \to \mathcal{A}^{2n-k}(X)$ is likewise defined point-wise.

(ii) The Hodge decomposition theorem asserts that $H^k(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$, and that each cohomology class in $H^{p,q}(X)$ can be uniquely represented by a *harmonic* (p,q)–form $\alpha \in \mathcal{A}^{p,q}(X)$.

(iii) The "Kähler identities" assert that the Laplacian commutes with $*$ and the $\mathfrak{sl}(2)$–triple $\{M, Y, N\}$. This means that they descend to well-defined operators on $H^\bullet(X,\mathbb{R})$. This gives $H^\bullet(X,\mathbb{R})$ the structure of an $\mathfrak{sl}(2)$–representation, and $H^\bullet(X,\mathbb{R})$ inherits the Lefschetz decomposition (Hard Lefschetz theorem).

(iv) One obtains a polarization on the primitive cohomology

$$H^k_{\mathrm{prim}}(X,\mathbb{R}) \ = \ \ker\{N : H^k(X,\mathbb{R}) \to H^{k-2}(X,\mathbb{R})\} \ \subset \ H^k(X,\mathbb{R})$$

by integrating Q_k over X (Hodge–Riemann bilinear relations).

For details, see [Huy05, Chap. 3].

Appendix A. Exterior Algebra

Let T be a real, finite dimensional vector space.

Definition A.1. A *k–tensor* is a k–linear map

$$\tau : T^* \times \cdots \times T^* \ \to \ \mathbb{R}\,.$$

Let $T^{\otimes k}$ denote the vector space of k–tensors.

Example A.2. One-tensors are just linear maps $T^* \to \mathbb{R}$. Two-tensors are bilinear forms. For example, an inner product on T^* is a 2–tensor.

Given $v_1, \ldots, v_k \in T$, let $v_1 \otimes \cdots \otimes v_k$ denote the k–tensor defined by

$$v_1 \otimes \cdots \otimes v_k(\mu_1, \ldots, \mu_k) \ = \ \mu_1(v_1) \cdots \mu_k(v_k)\,, \quad \forall\ \mu_i \in T^*\,.$$

Exercise A.3. If $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a basis of T, then $\{\varepsilon_{i_i} \otimes \cdots \otimes \varepsilon_{i_k}\}$ is a basis of $T^{\otimes k}$.

Definition A.4. Let $\mathfrak{S}_k$ be the permutation group on k–letters. Fix a basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$. Given $\sigma \in \mathfrak{S}_k$ we may define a linear map $\mathbb{R}^k \to \mathbb{R}^k$ by specifying $e_i \mapsto e_{\sigma(i)}$. The *sign* of σ is the determinant of this linear map, and is denoted $(-1)^\sigma$.

Definition A.5. A k–tensor τ is *alternating* if

$$\tau(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(k)}) \ = \ (-1)^\sigma \tau(\mu_1, \ldots, \mu_k)\,,$$

for all $\mu_i \in T^*$ and $\sigma \in \mathfrak{S}_k$. Let $\bigwedge^k T \subset T^{\otimes k}$ denote the vector subspace of alternating k–tensors.

Example A.6. Every 1–tensor is (trivially) alternating: $T = \bigwedge^1 T$. The alternating 2–tensors are the skew-symmetric forms: $\tau(\mu_1, \mu_2) = -\tau(\mu_2, \mu_1)$.

Example A.7. Let $\times$ denote the cross-product on $\mathbb{R}^3$, and fix $v \in \mathbb{R}^3$. Then

$$\tau_v(u_1, u_2) \;=\; (u_1 \times u_2) \cdot v$$

is an alternating 2–tensor on $T = (\mathbb{R}^3)^*$.

Example A.8. If we regard elements $x = (x^i) \in \mathbb{R}^n$ as column vectors, then a choice of n–elements $x_1, \ldots, x_n \in \mathbb{R}^n$ determines an $n \times n$ matrix (x^i_j), and

$$\delta(x_1, \ldots, x_n) \;=\; \det(x^i_j) \;=\; \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma x^i_{\sigma(j)}$$

is an alternating n–tensor.

Definition A.9. Given $v_1, \ldots, v_k \in T$, let $v_1 \wedge \cdots \wedge v_k$ be the k–tensor defined by

$$v_1 \wedge \cdots \wedge v_k \;=\; \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma v_\sigma(1) \otimes \cdots \otimes v_{\sigma(k)} \,.$$

Exercise A.10. Prove that $\{\varepsilon_{i_i} \wedge \cdots \wedge \varepsilon_{i_k} \mid 1 \le i_1 < \cdots < i_k \le \dim T\}$ is a basis of $\bigwedge^k T$.

Exercise A.11. Given $\alpha \in \bigwedge^k T$ and $\beta \in \bigwedge^\ell T$, prove that

$$\alpha \wedge \beta \;=\; (-1)^{k\ell} \beta \wedge \alpha \,.$$

Definition A.12. We call $\bigwedge^k T$ the *k–the exterior power of T*. By convention $\bigwedge^0 T = \mathbb{R}$. We call $\bigwedge^\bullet T = \oplus_k \bigwedge^k T$ the *exterior algebra*.

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