A trinity in affine Schubert calculus

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Peterson isomorphism in homology

G : semi-simple, simply connected algebraic group over $\mathbb{C}$

$$\rightsquigarrow G[\![z]\!] := G(\mathbb{C}[\![z]\!]) \subset G(\!(z)\!) := G(\mathbb{C}(\!(z)\!))$$

$T \subset B$: maximal torus of G $\subset$ maximal solvable subgroup of G

$\rightsquigarrow X := G/B$: flag variety, $\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!]$: affine Grassmannian

Theorem (Peterson 1997, Lam-Shimozono 2010)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$qH^{\bullet}_T(X)_{\mathrm{loc}} \cong H^T_{\bullet}(\mathrm{Gr}_G)_{\mathrm{loc}} \qquad (\diamondsuit).$$

1. The LHS is known to be described by (finite) Toda lattice (Givental, Kim);
2. The RHS is known to be described in Lie theoretic terms (Ginzburg);
3. Kostant described finite Toda lattice in terms of Lie algebras.

These three works were earlier (1995, $\sim$mid 1990s, 1970s), and hence the deepest insight of Peterson lie on the identification of bases

and the Peterson variety that we do not explain today.

Peterson isomorphism in homology

Theorem (Peterson 1997, Lam-Shimozono 2010)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$qH^{\bullet}_{T}(X)_{\mathrm{loc}} \cong H_{\bullet}^{T}(\mathrm{Gr}_G)_{\mathrm{loc}} \qquad (\diamondsuit).$$

The deepest aspect of the insight of Peterson lie on the identification of bases.

1. The only proof of $(\diamondsuit)$ (with the presence of Schubert basis) known today is to apply Mihalcea's Chevalley type formula (2007);
2. There are several explanations of $(\diamondsuit)$ including these from symplectic homology, but they are not enough to explain the coincidence of the bases.

The goal of this talk is to give a proof of the K-theoretic analogue of $(\diamondsuit)$.

As far as I know, this gives the first explanation as to why the Schubert bases correspond each other (through the whole business of this type),
in the sense that the identification comes before knowing precise numerical formulas.

Lam-Li-Mihalcea-Shimozono's conjecture

Conjecture (Lam-Li-Mihalcea-Shimozono 2018)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$qK_T(X)_{\mathrm{loc}} \cong K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \qquad (\spadesuit'),$$

where $qK_T(X)$ is the quantum K-group in the sense of Givental-Lee.

Let $N := [B, B]$. We have defined the semi-infinite flag manifolds as:

$$\mathbf{Q}^{\mathrm{rat}} := G(\!(z)\!)/T \cdot N(\!(z)\!)$$

We have the notion of equivariant K-group $K_{T \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$.

Theorem ("Trinity in affine Schubert calculus")

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{tikzcd} & K_T(\mathbf{Q}^{\mathrm{rat}}) & \\ K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \arrow[ur, hook, "\Phi"] \arrow[rr] & & qK_T(X)_{\mathrm{loc}} \arrow[ul, hook', "\Psi"'] \end{tikzcd} \qquad (\spadesuit).$$

Correspondence between labels

$W_{\mathrm{af}} = W \ltimes Q^\vee$: affine and finite Weyl group, and the coroot lattice of G
$Q^\vee_+, Q^\vee_\le$: $\mathbb{Z}_\ge$-span of $+$-coroots, antidominant coroots ($Q^\vee_+ \cap Q^\vee_\le = \{0\}$)
$\mathbf{I} \subset G[\![z]\!]$: the Iwahori subgroup of $G[\![z]\!]$

1. $qK_T(X) \doteq K_T(X) \otimes \mathbb{C}Q^\vee_+$ and $B \backslash X \leftrightarrow W$. It follows that

$$qK_T(X) \doteq \bigoplus_{w \in W, \beta \in Q^\vee_+} \mathbb{C}[T][\mathcal{O}^w]Q^\beta;$$

2. $\mathbf{I} \backslash \mathrm{Gr}_G \sim W \times Q^\vee_\le$. It follows that

$$K_T(\mathrm{Gr}_G) = \sum_{w \in W, \beta \in Q^\vee_\le} \mathbb{C}[T][\mathcal{O}^{\mathrm{Gr}}_{(w,\beta)}] \quad \text{almost direct sum;}$$

3. $\mathbf{I} \backslash \mathbf{Q}^{\mathrm{rat}} \leftrightarrow W_{\mathrm{af}}$. It follows that

$$K_{T \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \bigoplus_{v \in W_{\mathrm{af}}} \mathbb{C}[T \times \mathbb{G}_m][\mathcal{O}^v].$$

Here we intentionally neglected completions for the sake of simplicity.

Correspondence between labels

1. $qK_T(X) \doteq K_T(X) \otimes \mathbb{C}Q_+^\vee$ and $B\backslash X \leftrightarrow W$. It follows that

$$qK_T(X) \doteq \bigoplus_{w \in W, \beta \in Q_+^\vee} \mathbb{C}[T][\mathcal{O}^w]Q^\beta \qquad Q_+^\vee \cap Q_\leq^\vee = \{0\}.$$

2. $\mathbf{I}\backslash \mathrm{Gr}_G \sim W \times Q_<^\vee$. It follows that

$$K_T(\mathrm{Gr}_G) = \sum_{w \in W, \beta \in Q_<^\vee} \mathbb{C}[T][\mathcal{O}^{\mathrm{Gr}}_{(w,\beta)}] \qquad \text{almost direct sum}$$

3. $\mathbf{I}\backslash \mathbf{Q}^{\mathrm{rat}} \leftrightarrow W_{\mathrm{af}}$. It follows that

$$K_{T \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \bigoplus_{v \in W_{\mathrm{af}}} \mathbb{C}[T \times \mathbb{G}_m][\mathcal{O}^v].$$

In particular, if we enlarge $Q_+^\vee \subset Q^\vee$ and $Q_\leq^\vee \subset Q^\vee$ in the first two items via localization, then we find $\mathbb{C}[T]$-module (but not yet ring) "isomorphisms"

$$qK_T(X)_{\mathrm{loc}} \doteq K_T(\mathbf{Q}^{\mathrm{rat}}) \doteq K_T(\mathrm{Gr}_G)_{\mathrm{loc}}.$$

Localization of $K_T(\mathrm{Gr}_G)$

The product structure of $K_T(\mathrm{Gr}_G)$ is via the convolution product $\odot$, defined through the diagram

$$\mathrm{Gr} \times \mathrm{Gr} \leftarrow G(\!(z)\!) \times \mathrm{Gr} \rightarrow G(\!(z)\!) \times^{\mathbf{I}} \mathrm{Gr} \xrightarrow{\text{mult}} \mathrm{Gr}$$

Lemma (folklore, K)

For $\beta_1, \beta_2 \in Q^\vee_-$ and $w \in W$, we have $[\mathcal{O}^{\mathrm{Gr}}_{(w,\beta_1)}] \odot [\mathcal{O}^{\mathrm{Gr}}_{(e,\beta_2)}] = [\mathcal{O}^{\mathrm{Gr}}_{(w,\beta_1+\beta_2)}]$.

Corollary

For $\beta \in Q^\vee_\leq$, the $\odot$-action of the $[\mathcal{O}^{\mathrm{Gr}}_{(e,\beta)}]$ on $K_T(\mathrm{Gr}_G)$ is free. Adjoining their inverses yields $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$ whose basis is indexed by W_{af} through

$$W_{\mathrm{af}} \ni (w, \beta) \leftrightarrow [\mathcal{O}^{\mathrm{Gr}}_{(w,\beta_1)}] \odot [\mathcal{O}^{\mathrm{Gr}}_{(e,\beta_2)}]^{-1} \qquad \beta = \beta_1 - \beta_2.$$

In particular, the RHS does not depend on the choice of β_1, β_2.

The group $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ – recollections

We have

$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \subset \{ \sum_{w\in W_{\mathrm{af}}} a_w[\mathcal{O}^w] \mid a_w \in \mathbb{C}(\!(q^{-1})\!)[T], a_w \neq 0 \text{ only if } w \gg_{\frac{\infty}{2}} -\infty\},$$

where q is degree one $\mathbb{G}_m$-character, and $\geq_{\frac{\infty}{2}}$ is the closure ordering of **I**-orbits.

- $G(\!(z)\!)$-equivariant line bundles on $\mathbf{Q}^{\mathrm{rat}}$ are of the shape $\mathcal{O}(\lambda)$ ($\lambda \in P$), and each of them admits the $G(\!(z)\!) \ltimes \mathbb{C}^\times$-action, where $\mathbb{C}^\times$ is the loop rotation;
- Each $C \in K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ defines a $\mathbb{C}(\!(q^{-1})\!)[T]$-linear functional
$$P \ni \lambda \mapsto \chi_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}, C \otimes_{\mathcal{O}} \mathcal{O}(\lambda)) \in \mathbb{C}(\!(q^{-1})\!)[T]$$
modulo negligible elements;
- The translation $[\mathcal{O}^w] \mapsto [\mathcal{O}^{w(e,\beta)}]$ ($\beta \in Q^\vee \subset W_{\mathrm{af}}$) induces $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ the right $\mathbb{C}[\![Q^\vee]\!]$-module structure that further induces the right action of
$$\mathbb{C}(\!(Q^\vee)\!) := \mathbb{C}[Q^\vee] \otimes_{\mathbb{C}[Q^\vee_+]} \mathbb{C}[\![Q^\vee]\!];$$
- If we expand $[\mathcal{O}^w(\lambda)] = \sum_v a_v(\lambda)[\mathcal{O}^v]$, then $a_v(\lambda) \in \mathbb{C}[T \times \mathbb{G}_m]$;

The group $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ – implications

We have

$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \subset \{ \sum_{w\in W_{\mathrm{af}}} a_w[\mathcal{O}^w] \mid a_w \in \mathbb{C}(\!(q^{-1})\!)[T], a_w \neq 0 \text{ only if } w \gg_{\frac{\infty}{2}} -\infty\},$$

where q is degree one $\mathbb{G}_m$-character, and $\geq_{\frac{\infty}{2}}$ is the closure ordering of $\mathbf{I}$-orbits.

- Element of $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ is formal $\mathbb{C}(\!(q^{-1})\!)[T]$-linear combination of $[\mathcal{O}^w]$
- Topology of $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ is given by $\mathrm{Span}\{[\mathcal{O}^{(u,\beta)}]\}_{u\in W,\beta\geq\gamma}$ ($\gamma \in Q^\vee$);

Define subset $K'_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \subset K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ s.t. $a_w \in \mathbb{C}[T\times\mathbb{G}_m]$.

- We have $[\mathcal{O}(\lambda)]\otimes \circlearrowright K'_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ for $\lambda \in P$;
- The right $\mathbb{C}(\!(Q^\vee)\!)$-action also descends to $K'_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$.

$\rightsquigarrow$ The $q=1$ specialization $K_T(\mathbf{Q}^{\mathrm{rat}})$ of $K'_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ admits line bundle tensor product and translation actions.

Remarks on T-equivariant K-group of $\mathbf{Q}^{\mathrm{rat}}$

We have

$$K_T(\mathbf{Q}^{\mathrm{rat}}) \subset \{ \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}^w] \mid a_w \in \mathbb{C}[T], a_w \neq 0 \text{ only if } w \gg_{\frac{\infty}{2}} -\infty \},$$

on which line bundle tensor product and translation action exists.

- We have $[\mathcal{O}_{\mathbf{Q}^{\mathrm{rat}}}] \notin K_T(\mathbf{Q}^{\mathrm{rat}})$ since $\mathbf{Q}^{\mathrm{rat}}$ is not a Schubert variety of $\mathbf{Q}^{\mathrm{rat}}$ $\Leftarrow$ Each $\mathbf{I}$-orbit in $\mathbf{Q}^{\mathrm{rat}}$ has ∞-codim (as well as ∞-dim);
- In particular, $[\mathcal{O}(\lambda)] \otimes \circlearrowright K_T(\mathbf{Q}^{\mathrm{rat}})$ is not multiplication in $K_T(\mathbf{Q}^{\mathrm{rat}})$;
- If we restrict to $K_T(\mathbf{Q}(e))$, then $[\mathcal{O}(\lambda)] \otimes \equiv [\mathcal{O}_{\mathbf{Q}(e)}(\lambda)] \otimes$, and it is indeed realized as multiplication of $K_T(\mathbf{Q}(e))$ $\Leftarrow$ $\mathbf{Q}(e)$ is a scheme.

Back to Gr_G, we have indeed the same problem. In particular, the product $\odot$ on $H_\bullet(\mathrm{Gr}_G)$ a priori has nothing to do with the multiplication in cohomology.

This point was obscured since $H^\bullet(\mathrm{Gr}_G)$ is indeed the enveloping algebra of a commutative Lie algebra (Ginzburg, cf. Yun-Zhu).

A tiny example – tensoring with $\mathcal{O}(\varpi)$

When $G = SL(2)$, we have

$$\mathbf{Q}(e) = \mathbb{P}(\mathbb{C}^2[\![z]\!]) = \bigcup_{w \in \mathfrak{S}_2, m \geq 0} \mathbf{Q}(w, m\alpha^\vee),$$

where $(w, m\alpha^\vee) \in W_{\mathrm{af}}$. We have

$$H^i(\mathbf{Q}(w, m\alpha^\vee), \mathcal{O}(l\varpi)) \cong \begin{cases} S^l\left(\mathbb{C}^{1+\delta_{w,e}}\xi^m \oplus \mathbb{C}^2[\xi]\xi^{m+1}\right) & (i = 0, l \geq 0) \\ \{0\} & (\text{else}) \end{cases}$$

where ξ (or ξ^m) is the formal variable dual to z (or z^m). We have

$$\mathrm{ch}_T\, H^0(\mathbf{Q}(e), \mathcal{O}(\varpi)) \cong \mathrm{ch}_T\, \mathbb{C}^2[\xi] = \infty(e^\varpi + e^{-\varpi}).$$

Nevertheless, we can interpret as

$$[\mathcal{O}(\varpi)] \otimes [\mathcal{O}_{\mathbf{Q}(e)}] = \sum_{w \in \mathfrak{S}_2, m \geq 0} q^{-m} e^{-w\varpi} [\mathcal{O}_{\mathbf{Q}(w, m\alpha^\vee)}] \quad \in K_{T \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$$

by examining the $(T \times \mathbb{G}_m)$-characters.

- The sum here is infinite $\rightsquigarrow$ $[\mathcal{O}(\varpi)]\otimes$ needs completion to be well-defined.

Another tiny example – tensoring with $\mathcal{O}(-\varpi)$

When $G = SL(2)$, we have

$$\mathbf{Q}(e) = \mathbb{P}(\mathbb{C}^2[\![z]\!]) = \bigcup_{w \in \mathfrak{S}_2, m \ge 0} \mathbf{Q}(w, m\alpha^\vee) \quad \cong \mathbb{P}^\infty,$$

where $(w, m\alpha^\vee) \in W_{\mathrm{af}}$. Thus, we have finite expressions

$$[\mathcal{O}(-\varpi)] \otimes [\mathcal{O}_{\mathbf{Q}(e,m\alpha^\vee)}] = e^{\varpi}([\mathcal{O}_{\mathbf{Q}(e,m\alpha^\vee)}] - [\mathcal{O}_{\mathbf{Q}(s,m\alpha^\vee)}])$$
$$[\mathcal{O}(-\varpi)] \otimes [\mathcal{O}_{\mathbf{Q}(s,m\alpha^\vee)}] = e^{-\varpi}([\mathcal{O}_{\mathbf{Q}(e,m\alpha^\vee)}] - q^{-1}[\mathcal{O}_{\mathbf{Q}(e,(m+1)\alpha^\vee)}]),$$

where the second term is the (unique) codimension one orbit. Thus, $[\mathcal{O}(-\varpi)]\otimes$ makes sense on $K_{T \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ without completion.

- The above formula ($q = 1$) coincides the multiplication rule of $qK_T(X)_{\mathrm{loc}}$ and $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$ (with respect to their Schubert basis);
- The action $[\mathcal{O}(-\varpi)]\otimes$ corresponds to $[\mathcal{O}_X(-\varpi)]\star$ in $qK_T(X)_{\mathrm{loc}}$. We have

  $$([\mathcal{O}(-\varpi)]\star)^{-1} \neq [\mathcal{O}_X(\varpi)]\star,$$

  and the former action on $qK_T(X)$ is infinite (the latter is finite)
  $\rightsquigarrow$ it does not act on the polynomial version $K_T(X) \otimes \mathbb{C}[Q_+^\vee] \subset qK_T(X)$.

Inclusion $K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \subset K_T(\mathbf{Q}^{\mathrm{rat}})$

For each simple root α of $G((z))$, we have $\mathbf{I} \subset \mathbf{I}(\alpha) \subset G((z))$ s.t.

$$\mathbf{I}(\alpha)/\mathbf{I} \cong \mathbb{P}^1 \qquad\qquad \text{(minimal parabolic).}$$

The multiplication induces morphisms

$$\mathbf{I}(\alpha) \times^{\mathbf{I}} \mathrm{Gr}_G \to \mathrm{Gr}_G \qquad \mathbf{I}(\alpha) \times^{\mathbf{I}} \mathbf{Q}^{\mathrm{rat}} \to \mathbf{Q}^{\mathrm{rat}},$$

that induces an action of (level zero) nil-DAHA $\mathscr{H}^{\mathrm{nil}}$ on $K_T(\mathrm{Gr}_G)$ and $K_T(\mathbf{Q}^{\mathrm{rat}})$.

Proposition (Kostant-Kumar, K)

1. $\mathscr{H}^{\mathrm{nil}} \circlearrowright K_T(\mathrm{Gr}_G)$ extends to $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$, and commutes with $\odot[\mathcal{O}^{\mathrm{Gr}}_{(e,\beta)}]^{\pm 1}$;
2. $\mathscr{H}^{\mathrm{nil}} \circlearrowright K_T(\mathbf{Q}^{\mathrm{rat}})$ commutes with the twist of $\mathbf{I}$-orbit labels by $Q^\vee \subset W_{\mathrm{af}}$.

The above two $\mathscr{H}^{\mathrm{nil}} \otimes \mathbb{C}Q^\vee$-modules are cyclic. In addition, $[\mathcal{O}^{\mathrm{Gr}}_{(w,\beta)}] \mapsto [\mathcal{O}^{(w,\beta)}]$ extends to an embedding $K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \subset K_T(\mathbf{Q}^{\mathrm{rat}})$ as modules.

We can write down the line bundle twist of $K_T(\mathbf{Q}^{\mathrm{rat}})$ in terms of the $\odot$-action on $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$. The formula is pretty simple for primitive anti-nef line bundles (p12).

Quantum K-group

The quantum K-group $qK_T(X)$ is a vector space $K_T(X) \otimes \mathbb{C}[\![Q^\vee]\!]$ equipped with a linear functional

$$qK_T(X)^{\otimes 3} \ni (a, b, c) \mapsto \langle a, b, c \rangle_{\mathrm{GW}} = \sum_{\beta \in Q_+^\vee} Q^\beta \langle a, b, c \rangle_{\mathrm{GW}}^\beta \in \mathbb{C}[T][\![Q_+^\vee]\!]$$

with the following properties:

1. $\beta \in Q_+^\vee \subset H_2(X, \mathbb{Z})$ is understood to be a class of algebraic curves on X;
2. Q^β $(\beta \in Q_+^\vee)$ is a formal invariant and hence the functional is linear with respect to the $\mathbb{C}[\![Q_+^\vee]\!]$-actions;
3. $\langle a, b, c \rangle_{\mathrm{GW}}$ is symmetric with respect to the $\mathfrak{S}_3$-permutation.

The quantum K-product $\star$ is defined through

$$\langle a \star b, c, [\mathcal{O}_X] \otimes 1 \rangle_{\mathrm{GW}} = \langle a, b, c \rangle_{\mathrm{GW}} \qquad a, b, c \in qK_T(X).$$

Theorem (Givental, Lee)

The binary operation $\star$ defines $qK_T(X)$ with a commutative and associative algebra structure with $1 = [\mathcal{O}_X] \equiv [\mathcal{O}_X] \otimes 1$.

Three-point invariants

For each $\beta \in Q_+^\vee$, there exists a space $X_{\beta,3}$ whose points parametrizes:

1. a connected curve C with genus zero with three distinct points $p_1, p_2, p_3 \in C$;
2. the singularity of C is mild, and p_1, p_2, p_3 are not singular points of C;
3. $f : C \to X$ define $f_*[C] = \beta \in H_2(X, \mathbb{Z})$;
4. the automorphism group of the data (C, p_1, p_2, p_3, f) is finite.

Theorem (Fulton-Phandharipande)

The variety $X_{\beta,3}$ (and $\mathfrak{X}_{\beta,k}$ defined later) exists, and it is a proper normal algebraic variety with quotient singularities (smooth viewed as an algebraic stack).

Assignment $(C, p_1, p_2, p_3, f) \mapsto f(p_i) \in X \rightsquigarrow \mathrm{ev}_i : X_{\beta,3} \to X$ $(i = 1, 2, 3)$.

For $a, b, c \in K_T(X)$, we define

$$\langle a, b, c\rangle_{\mathrm{GW}}^{\beta} := \chi_T(X_{\beta,3}, \mathrm{ev}_1^* a \otimes \mathrm{ev}_2^* b \otimes \mathrm{ev}_3^* c) \in \mathbb{C}[T] \subset \mathbb{C}[T][\![Q_+^\vee]\!].$$

We extend $\langle \bullet, \bullet, \bullet\rangle_{\mathrm{GW}}^{\beta}$ to $qK_T(X)$ by the $\mathbb{C}[\![Q_+^\vee]\!]$-linearity.

Kollar's vanishing theorem

The Euler-Poincaré characteristics on $X_{\beta,3}$ is not necessarily easy to compute. For example, we have

$$\dim\left(X_{\beta,3}\right)^T \to \infty \quad \text{as} \quad \beta \to \infty.$$

Theorem (Kollar)

Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a proper morphism such that $\mathfrak{Y}, \mathfrak{X}$ has rational singularities and the general fiber F of f satisfies $H^i(F, \mathcal{O}_F) \cong \mathbb{C}^{\delta_{i0}}$. Then, for each locally free sheaf $\mathcal{M}$ on $\mathfrak{X}$, we have

$$H^\bullet(\mathfrak{X}, \mathcal{M}) = H^\bullet(\mathfrak{Y}, f^*\mathcal{M}).$$

$\rightsquigarrow$ replace $X_{\beta,3}$ with other varieties with rational singularities (connected by proper morphisms in two steps) to compute the Euler-Poincaré characteristics of some specific sheaves.

Graph spaces

For each $\beta \in Q_+^\vee$, there exists a space $\mathfrak{X}_{\beta,k}$ $(k = 0, 1, 2 \dots)$ whose points parametrizes:

1. a connected curve C with genus zero with k distinct points $p_1, \dots, p_k \in C$;
2. the singularity of C is mild, and $p_1, \dots, p_k$ are not singular points of C;
3. $f : C \to \mathbb{P}^1 \times X$ define $f_*[C] = (1, \beta) \in H_2(\mathbb{P}^1 \times X, \mathbb{Z}) = \mathbb{Z} \oplus H_2(X, \mathbb{Z})$;
4. the automorphism group of the data $(C, p_1, \dots, p_k, f)$ is finite.

Forgetting points or $\mathbb{P}^1$ yields maps

$$\mathfrak{X}_{\beta,0} \xleftarrow{\pi} \mathfrak{X}_{\beta,3} \xrightarrow{\psi} X_{\beta,3}$$

Since the general fiber of ψ is $\mathbb{P}^3$, Kollar's theorem yields

$$\langle a, b, c\rangle_{\mathrm{GW}}^{\beta} = \chi_T(\mathfrak{X}_{\beta,3}, \mathrm{ev}_1^* a \otimes \mathrm{ev}_2^* b \otimes \mathrm{ev}_3^* c) \in \mathbb{C}[T] \subset \mathbb{C}[T][\![Q_+^\vee]\!].$$

Moreover, we may have the extra $\mathbb{G}_m$-action acting on the first component of $\mathbb{P}^1 \times X$, that yields a q-deformation $\langle a, b, c\rangle_{\mathrm{GW}}^{q}$ of $\langle a, b, c\rangle_{\mathrm{GW}}$.

Evaluations on graph spaces

For each $\beta \in Q_+^\vee$, there exists a space $\mathfrak{X}_{\beta,k}$ ($k = 0, 1, 2 \ldots$) whose points parametrizes:

- $f : C \to \mathbb{P}^1 \times X$ define $f_*[C] = (1, \beta) \in H_2(\mathbb{P}^1 \times X, \mathbb{Z}) = \mathbb{Z} \oplus H_2(X, \mathbb{Z})$.

We have

$$\langle a, b, c \rangle_{\mathrm{GW}}^{\beta, q} = \chi_{T \times \mathbb{G}_m}(\mathfrak{X}_{\beta,3}, \mathrm{ev}_1^* a \otimes \mathrm{ev}_2^* b \otimes \mathrm{ev}_3^* c) \in \mathbb{C}[T \times \mathbb{G}_m].$$

If we examine the definition, then we find $a, b, c \in K_{T \times \mathbb{G}_m}(\mathbb{P}^1 \times X)$. We may take $a, b, c \in K_T(X)$, and promote to $K_{T \times \mathbb{G}_m}(\mathbb{P}^1 \times X)$ by

$$a := [\mathbb{C}_0] \boxtimes a, b := [\mathbb{C}_\infty] \boxtimes b, c := [\mathcal{O}_{\mathbb{P}^1}] \boxtimes c \qquad \text{(BASIC notation)}.$$

This is the same as restricting to $\mathfrak{X}^\flat_{\beta,k} \subset \mathfrak{X}_{\beta,k}$ such that $p_1 \mapsto 0, p_2 \mapsto \infty$ in $\mathbb{P}^1$. Buch-Chaput-Mihalcea-Perrin says that $\mathfrak{X}^\flat_{\beta,k}$ has rational singularities, and hence the (equivariant) Euler-Poincaré characteristic does not change by this procedure.

Quasi-map spaces

For $(C, p_1, \ldots, p_k, f) \in \mathfrak{X}_{\beta,k}$, we have a specific irreducible component $\mathbb{P}^1 \subset C$ that maps isomorphically to the first component of $\mathbb{P}^1 \times X$ using degree $(1, \beta)$.

Identifying different choices of $p_1, \ldots, p_k$ and $f|_{C \setminus \mathbb{P}^1}$, we obtain a set $\mathcal{Q}(\beta)$. This is a topological compactification of space of holomorphic maps $\mathbb{P}^1 \to X$ of degree β.

Theorem (Givental's main lemma; LLY, FFKM, DP)

The space $\mathcal{Q}(\beta)$ is an algebraic variety and the map $\mathfrak{X}_{\beta,0} \to \mathcal{Q}(\beta)$ is a map of algebraic varieties.

Theorem (K)

We have an identification $\mathcal{Q}(\beta) \cong \mathcal{Q}(0, \beta) \subset \mathbf{Q}^{\mathrm{rat}}$. In addition, $\mathcal{Q} := \bigcup_{\beta \in Q_+^\vee} \mathcal{Q}(\beta)$ is Zariski dense in an $\mathbf{I}$-orbit closure $\mathbf{Q}(e) \subset \mathbf{Q}^{\mathrm{rat}}$.

$\rightsquigarrow$ Each λ defines a line bundle on $\mathcal{Q}(\beta)$ and $\mathfrak{X}_{\beta,k}$ that we denote by $\mathcal{O}(\lambda)$.

$$(\lambda \in P, \text{ with } P \text{ the set of weights of } T)$$

Shift operators for line bundles

Theorem (Iritani-Milanov-Tonita)

For $\lambda \in P$, there is an endomorphism $A^\lambda(q)$ of $\mathbb{C}(\!(q^{-1})\!) \otimes qK_T(X)$ such that

$$\left\langle a, A^\lambda(q)b, [\mathcal{O}_X] \right\rangle_{\mathrm{GW}}^{q,\beta} = \chi_{T \times \mathbb{G}_m}(\mathfrak{X}^\flat_{\beta,2}, \mathrm{ev}_1^* a \otimes \mathrm{ev}_2^* b \otimes \mathcal{O}(\lambda))$$

for every $\beta \in Q_+^\vee$ and $a, b \in qK_T(X)$. In addition, $A^\lambda(1)$ is the multiplication of an element of $qK_T(X)$.

This theorem requires the localization theorem, applied to the $\mathbb{G}_m$-action on $\mathfrak{X}_{\beta,k}$.

Observation (easy by looking at the classical limit)

The elements $A^\lambda(1) \in qK_T(X)$ $(\lambda \in P)$, generate $qK_T(X)$ as $\mathbb{C}[T][\![Q_+^\vee]\!]$-algebra.

$\rightsquigarrow$ If we know the relations satisfied by $\{A^\lambda(1)\}_\lambda$, then we know the ring $qK_T(X)$. But we proceed slightly different way.

Inclusion $qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$

Theorem (K)

There exists an injective $\mathbb{C}[T]$-algebra map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$ with the following properties:

1. We have $\Psi(A^\lambda(1)\bullet) = [\mathcal{O}(\lambda)] \otimes \Psi(\bullet)$ for each $\lambda \in P$.
2. It sends $[\mathcal{O}^w]Q^\beta$ to $[\mathcal{O}^{(w,\beta)}]$ for $w \in W, \beta \in Q_+^\vee$.

Corollary

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{tikzcd} & K_T(\mathbf{Q}^{\mathrm{rat}}) & \\ K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \arrow[ur, hook, "\Phi"] \arrow[rr] & & qK_T(X)_{\mathrm{loc}} \arrow[ul, hook', "\Psi"'] \end{tikzcd} \qquad (\spadesuit).$$

This also equips $K_T(\mathbf{Q}^{\mathrm{rat}})$ with the structure of an algebra with $1 = [\mathcal{O}_e]$.

Remarks on the triangle (♠)

Corollary

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{tikzcd} & K_T(\mathbf{Q}^{\mathrm{rat}}) & \\ (K_T(\mathrm{Gr}_G)_{\mathrm{loc}}, \odot) \arrow[ur, hook, "\Phi"] \arrow[rr] & & (qK_T(X)_{\mathrm{loc}}, \star) \arrow[ul, hook', "\Psi"'] \end{tikzcd} \qquad (\spadesuit).$$

This also equips $K_T(\mathbf{Q}^{\mathrm{rat}})$ with the structure of an algebra with $1 = [\mathcal{O}_e]$.

- The diagram as based $\mathbb{C}[T][Q^\vee]$-modules is already explained;
- The effect of $e^{-\varpi_i}[\mathcal{O}(-\varpi_i)]\otimes$ on $K_T(\mathbf{Q}^{\mathrm{rat}})$ yields

  $$(1 - [\mathcal{O}^{\mathrm{Gr}}_{(s_i,\beta)}][\mathcal{O}^{\mathrm{Gr}}_{(e,\beta)}]^{-1}) \odot \text{ on } K_T(\mathrm{Gr}_G)_{\mathrm{loc}}, \text{ and } (1 - [\mathcal{O}^s_i]) \star \text{ on } qK_T(X)$$

  This identifies the multiplications of $[\mathcal{O}^{\mathrm{Gr}}_{(s_i,\beta)}][\mathcal{O}^{\mathrm{Gr}}_{(e,\beta)}]^{-1}$ and $[\mathcal{O}^{s_i}]$.
- A consequence of (♠) is the finiteness of $\star$;

Remarks on the triangle (♠)

Corollary

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{tikzcd}[column sep=small] & K_T(\mathbf{Q}^{\mathrm{rat}}) & \\ (K_T(\mathrm{Gr}_G)_{\mathrm{loc}}, \odot) \arrow[ur, hook, "\Phi"] \arrow[rr] & & (qK_T(X)_{\mathrm{loc}}, \star) \arrow[ul, hook', "\Psi"'] \end{tikzcd} \qquad (\spadesuit).$$

This also equips $K_T(\mathbf{Q}^{\mathrm{rat}})$ with the structure of an algebra with $1 = [\mathcal{O}_e]$.

- A consequence of (♠) is the finiteness of $\star$, i.e. $\star$-product of two Schubert cells are finite linear combination of Schubert cells;
- In particular, if $[\mathcal{O}_{\mathbf{Q}(e)}(\lambda)]$ $(\lambda \in P)$ is the finite sum with respect to the Schubert classes, then $A^\lambda(q)$ is finite. This happen precisely if $-\lambda \in P_+$;
- Therefore, the quantum multiplication $[\mathcal{O}(-\varpi_i)]\star$ has inverse in $qK_T(X)$, but not in the polynomial version $K_T(X) \otimes \mathbb{C}Q_+^\vee \subset qK_T(X)$ (P12);
- $([\mathcal{O}(-\varpi_i)]\star)^{-1}$ corresponds to $[\mathcal{O}(\varpi_i)]\otimes$ on $K_T(\mathbf{Q}^{\mathrm{rat}})$, that is essential in the construction of Ψ. $\rightsquigarrow$ our versions of $K_T(\mathbf{Q}^{\mathrm{rat}})$ and $qK_T(X)$ are good

Construction of the map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$

We compare a bit precise $\mathbb{G}_m$-equivariant version (i.e. q alive). We have presentations (as modules)

$$qK_{T\times\mathbb{G}_m}(X) = \mathbb{C}[T\times\mathbb{G}_m][A^{\lambda}(q); \lambda \in P][\![Q^{\vee}]\!]/\sim$$
$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \mathbb{C}[T\times\mathbb{G}_m][[\mathcal{O}(\lambda)]\otimes; \lambda \in P](\!(Q^{\vee})\!)/\sim,$$

where the generators are $[\mathcal{O}_X] \in qK_{T\times\mathbb{G}_m}(X)$, $[\mathcal{O}^e] \in K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$, and $\sim$ represents modding out the defining relations.

Warning: we pretend the generators are commutative, but in fact it is not (the non-commutativity is mild and does not harm the logic).

Assume that $(\sum_{\lambda,\beta} a_{\lambda,\beta} A^{\lambda}(q) Q^{\beta})1 = 0$ in $qK_{T\times\mathbb{G}_m}(X)$ with $a_{\lambda,\beta} \in \mathbb{C}[T\times\mathbb{G}_m]$. We have

$$\left\langle (\sum_{\lambda,\beta} a_{\lambda} A^{\lambda}(q) Q^{\beta})[\mathcal{O}_X], [\mathcal{O}_X], [\mathcal{O}_X] \right\rangle^{q}_{\mathrm{GW}} \equiv 0.$$

Map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$, cont'd

We compare a bit precise $\mathbb{G}_m$-equivariant version (i.e. q alive). We have presentations (as modules)

$$qK_{T\times\mathbb{G}_m}(X) \doteq \mathbb{C}[T\times\mathbb{G}_m][A^\lambda(q); \lambda \in P][\![Q^\vee]\!]/\sim$$
$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \mathbb{C}[T\times\mathbb{G}_m][[\mathcal{O}(\lambda)]\otimes; \lambda \in P](\!(Q^\vee)\!)/\sim .$$

Assume that $(\sum_{\lambda,\beta} a_{\lambda,\beta} A^\lambda(q) Q^\beta)1 = 0$ in $qK_{T\times\mathbb{G}_m}(X)$ with $a_{\lambda,\beta} \in \mathbb{C}[T\times\mathbb{G}_m]$. We have

$$\left\langle (\sum_{\lambda,\beta} a_{\lambda,\beta} A^\lambda(q) Q^\beta)[\mathcal{O}_X], [\mathcal{O}_X], [\mathcal{O}_X] \right\rangle^q_{\mathsf{GW}} \equiv 0.$$

expanding this yields

$$\begin{aligned}0 &= \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{-\langle\lambda,\beta\rangle} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,3}, \mathcal{O}(\lambda) \otimes \bigotimes_{i=1}^{3} \mathrm{ev}_i^*[\mathcal{O}_X]) \\ &= \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,0}, \mathcal{O}(\lambda)) = \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathcal{Q}(\gamma), \mathcal{O}(\lambda)),\end{aligned}$$

where the last equality requires $\mathcal{Q}(\beta)$ to have rational singularities.

Map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$, cont'd

We compare a bit precise $\mathbb{G}_m$-equivariant version (i.e. q alive). We have presentations (as modules)

$$qK_{T\times\mathbb{G}_m}(X) \doteq \mathbb{C}[T\times\mathbb{G}_m][A^\lambda(q);\lambda\in P][\![Q^\vee]\!]/\sim$$

$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \mathbb{C}[T\times\mathbb{G}_m][[\mathcal{O}(\lambda)]_{\otimes_3};\lambda\in P](\!(Q^\vee)\!)/\sim.$$

$$0=\sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{-\langle\lambda,\beta\rangle} Q^{\beta+\gamma}\chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,3},\mathcal{O}(\lambda)\otimes\bigotimes_{i=1}\mathrm{ev}_i^*[\mathcal{O}_X])$$

$$=\sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma}\chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,0},\mathcal{O}(\lambda))=\sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma}\chi_{T\times\mathbb{G}_m}(\mathcal{Q}(\gamma),\mathcal{O}(\lambda)),$$

where the last equality requires $\mathcal{Q}(\beta)$ to have rational singularities.

If we have $(\sum_{\lambda,\beta} a_{\lambda,\beta}A^\lambda(q)Q^\beta)1=0$, then we have $(\sum_{\lambda,\beta} a_{\lambda,\beta}A^{\lambda+\mu}(q)Q^\beta)1=0$ for every $\mu\in P$. In particular, the above equality yields

$$0=\sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma}\chi_{T\times\mathbb{G}_m}(\mathcal{Q}(\gamma),\mathcal{O}(\lambda))\qquad \mu\in P_+,$$

where the restriction $P_+\subset P$ indicates that we are interested in non-negligible change as linear functionals.

Map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$, cont'd

We compare a bit precise $\mathbb{G}_m$-equivariant version (i.e. q alive). We have presentations (as modules)

$$qK_{T\times\mathbb{G}_m}(X) \doteq \mathbb{C}[T\times\mathbb{G}_m][A^\lambda(q); \lambda \in P][\![Q^\vee]\!]/\sim$$

$$K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \doteq \mathbb{C}[T\times\mathbb{G}_m][[\mathcal{O}(\lambda)]_{\otimes}; \lambda \in P](\!(Q^\vee)\!)/\sim .$$

$$0 = \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{-\langle\lambda,\beta\rangle} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,3}, \mathcal{O}(\lambda) \otimes \bigotimes_{i=1}^{3} \mathrm{ev}_i^*[\mathcal{O}_X])$$

$$= \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathfrak{X}_{\gamma,0}, \mathcal{O}(\lambda)) = \sum_{\lambda,\beta,\gamma} a_{\lambda,\beta} q^{\bullet} Q^{\beta+\gamma} \chi_{T\times\mathbb{G}_m}(\mathcal{Q}(\gamma), \mathcal{O}(\lambda)),$$

where the last equality requires $\mathcal{Q}(\beta)$ to have rational singularities.

The coefficient of Q^κ for $\kappa \to \infty$ yields an identity

$$0 = \sum_{\lambda\in P, \beta\in Q^\vee_+} a_{\lambda,\beta} \chi_{T\times\mathbb{G}_m}(\mathbf{Q}(\beta), \mathcal{O}(\lambda+\mu)) \qquad \forall \mu \in P_+.$$

This implies $\sum_{\lambda,\beta} a_{\lambda,\beta}[\mathcal{O}^{(e,\beta)}(\lambda)] = 0 \in K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$. $\rightsquigarrow$ Ψ is well-defined.

Inclusion $qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$, cont'd

Theorem (K)

There exists an injective $\mathbb{C}[T]$-algebra map $\Psi : qK_T(X) \hookrightarrow K_T(\mathbf{Q}^{\mathrm{rat}})$ with the following properties:

1. We have $\Psi(A^\lambda(1)\bullet) = [\mathcal{O}(\lambda)] \otimes \Psi(\bullet)$ for each $\lambda \in P$.
2. It sends $[\mathcal{O}^w]Q^\beta$ to $[\mathcal{O}^{(w,\beta)}]$ for $w \in W, \beta \in Q_+^\vee$.

Once we know Ψ is well-defined, then the size comparison forces it to be an injection.

In order to show the preservation of the bases, the above arguments tell us that we only need to show

$$\chi_{T\times\mathbb{G}_m}(\mathfrak{X}^\flat_{\beta,1}, \mathcal{O}(\lambda) \otimes \mathrm{ev}_1^*[\mathcal{O}^w]) = \chi_{T\times\mathbb{G}_m}(\mathcal{Q}(\beta) \cap \mathbf{Q}(w), \mathcal{O}(\lambda))$$

since the $\beta \to \infty$ in RHS yields $\chi_{T\times\mathbb{G}_m}(\mathbf{Q}(w), \mathcal{O}(\lambda))$. This follows if

- $\mathcal{Q}(\beta) \cap \mathbf{Q}(w)$ has rational singularities.

In our previous talk, we asserted that it is at least normal.

A proof of the rationality of singularities

We have two proofs (this one works in general, and another have some restriction).

Theorem (K)

There exists a finite stratification of $\mathcal{Q}(\beta) \cap \mathbf{Q}(w)$ such that each stratum have a local transversal slice (i.e. product decomposition of local rings), along $\mathcal{Q}(\beta)$.

If each of these transversal slices turned out to have rational singularities, then the non-rational singularities on that stratum extends to $\mathcal{Q}(\beta)$, that is a contradiction.

The local slice for $\mathcal{Q}(\beta - \gamma) \subset \mathcal{Q}(\beta)$ is constructed as follows: Use G-action to restrict to the locus where the map at $\infty \in \mathbb{P}^1$ is defined and lands on a particular point on $\mathbb{P}^1$. Such subspaces yields an inclusion $\mathcal{Z}(\beta - \gamma) \subset \mathcal{Z}(\beta)$ of the based map spaces (a.k.a) Zastava spaces. It satisfies the factorization property:

Theorem (Finkelberg-Mirković, Braverman-Finkelberg-Gaitsgory-Mirković)

There exists a map $\pi^\beta : \mathcal{Z}(\beta) \to \mathbb{A}^{2|\beta|}$ such that the pullback of some open subset of $\mathbb{A}^{2|\beta|}$ decomposes into the product of open subsets of $\mathcal{Z}(\beta - \gamma) \times \mathcal{Z}(\gamma)$.

Using the factorization map (and its twists), we can construct (many) local transversal slices of $\mathcal{Z}(\beta - \gamma) \subset \mathcal{Z}(\beta)$ that are enough to cover each stratum.

Remarks

1. Our considerations asserts that the q-shift operator $A^{-\varpi_i}(q)$ is (also) finite, that was one of the key ingredients in the proof of the finiteness of $qK(G/P)$ proved by Anderson-Chen-Tseng-Iritani;
2. Our results also describe the parabolic version of the Peterson isomorphism, that is very simple though little strange (arXiv:1906.09343);
3. It resolves the Finkelberg-Tsymbaliuk's conjecture about the relationship between $K_G(\mathrm{Gr}_G)$ and $K_L(\mathrm{Gr}_L)$, where $L \subset G$ is Levi (arXiv:2008.01310);
4. Our K-theoretic version of the Peterson isomorphism still not succeeded to recover the homology version. One reason: $H^{\mathrm{odd}}(\mathcal{Q}(\beta), \mathbb{C})$ is large.

Final Remark

We hope to expand the triangle (♠) by throwing in related concepts from geometry, representation theory, and number theory, as well as polishing each item.

Any suggestions/comments are welcome, and

Thank you very much for your attention!