Evolution equations

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Examples

Linear evolution equations:

- the Schrödinger equation on Euclidean spaces:

$$i\partial_t u + \Delta u = 0, \qquad u(0) = \phi;$$

- the heat equation on Euclidean spaces:

$$\partial_t u - \Delta u = 0, \qquad u(0) = \phi;$$

- the wave equation on Euclidean spaces:

$$\partial_t^2 u - \Delta u = 0, \qquad u(0) = \phi_0, \, \partial_t u(0) = \phi_1.$$

- The linear equations can be solved explicitly using the Fourier transform, for example for the Schrödinger equation

$$u(t) = e^{it\Delta}\phi, \qquad \widehat{u}(\xi, t) = e^{-it|\xi|^2}\widehat{\phi}(\xi).$$

Examples

Semilinear evolution equations:

- the pure power NLS: $u : \mathbb{R}^d \times [0, T] \to \mathbb{C}$,

$$i\partial_t u + \Delta u = \pm u|u|^{2p}, \qquad u(0) = \phi.$$

- the KdV equation: $u : \mathbb{R} \times [0, T] \to \mathbb{R}$,

$$\partial_t u + \partial_x^3 u = u\partial_x u, \qquad u(0) = \phi.$$

- the Schrödinger maps equation $u : \mathbb{R}^d \times [0, T] \to \mathbb{S}^2$,

$$\partial_t u = u \times \Delta u, \qquad u(0) = \phi.$$

Examples

The Navier-Stokes equations on Euclidean spaces:
$u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$

$$\partial_t u - \Delta u + (u \cdot \nabla u) + \nabla p = 0, \qquad \mathrm{div}\, u = 0,$$
$$u(0) = \phi.$$

Explicitly, if $u = (u_1, \ldots, u_d)$ then

$$\partial_t u_k - \Delta u_k + u_j \partial_j u_k + \partial_k p = 0, \qquad \partial_j u_j = 0,$$
$$u(0) = \phi.$$

Leray formulation: take divergence of the equation to solve for the pressure

$$-\Delta p = \partial_j \partial_k (u_j u_k).$$

Examples

Quasilinear evolution equations:

- The Euler equations: $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$

$$\partial_t u + (u \cdot \nabla u) + \nabla p = 0, \qquad \mathrm{div}\, u = 0,$$
$$u(0) = \phi.$$

- The Einstein-vacuum equations of General Relativity: $\mathbf{g}$ Lorentzian metric in an open set,

$$\mathbf{Ric}(\mathbf{g}) = 0.$$

In local coordinates this is a coupled system of wave equations for the metric components

$$\widetilde{\square}_{\mathbf{g}} \mathbf{g}_{\alpha\mu} = \partial_\alpha \mathbf{\Gamma}_\mu + \partial_\mu \mathbf{\Gamma}_\alpha + F^{\geq 2}_{\alpha\mu}(g, \partial g),$$

where $\widetilde{\square}_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ denotes the reduced wave operator. In *wave coordinates* $\mathbf{\Gamma}_\alpha = 0$ this becomes a quasilinear system of wave equations for the metric components.

We recall the Leray formulation of the Navier-Stokes equations

$$\begin{aligned} &\partial_t u_k - \Delta u_k = \mathcal{N}_k(u), \\ &\mathcal{N}_k(u) = -\partial_a(u_a u_k) - \partial_k(R_a R_b(u_a u_b)), \end{aligned} \tag{1}$$

where $R_a = |\nabla|^{-1}\partial_a$ denote the Riesz transforms.

Theorem: (local well-posedness)
Assume $\phi \in H^\rho(\mathbb{R}^d)$, $\rho > d/2$, satisfies $\|\phi\|_{H^\rho} < R$ and the divergence-free condition $\partial_j \phi_j = 0$. Then there is $T = T(R) > 0$ and a unique solution $u \in C([0,T] : H^\rho)$ of the equation (1), which is divergence-free $\partial_j u_j(x,t) = 0$.

Moreover, the flow map $\phi \to u$ is a continuous map from the ball of radius R in $H^\rho(\mathbb{R}^d)$ to the ball of radius $2R$ in $C([0,T] : H^\rho)$.

Local well-posedness: fixed-point argument

We rewrite the equation (1) in integral form (Duhamel formula)

$$u(t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}\mathcal{N}(u(s))\,ds.$$

We would like to construct the solution by the recursive scheme

$$u^{(n+1)}(t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}\mathcal{N}(u^{(n)}(s))\,ds,$$
$$u^{(0)}(t) = e^{t\Delta}\phi.$$

The procedure converges if

$$\Big\| \int_0^t e^{(t-s)\Delta}\mathcal{N}(f(s))\,ds - \int_0^t e^{(t-s)\Delta}\mathcal{N}(g(s))\,ds \Big\|_{L^\infty_T H^\rho} \ll \|f-g\|_{L^\infty_T H^\rho} \tag{2}$$

for any $f, g \in C([0,T] : H^\rho)$ with $\|f\|_{L^\infty_T H^\rho}, \|g\|_{L^\infty_T H^\rho} \leq 2R$.

Local well-posedness: fixed-point argument

Recall that H^ρ is an algebra, $\rho > d/2$, and

$$\mathcal{N}_k(u) = -\partial_a(u_a u_k) - \partial_k(R_a R_b(u_a u_b)),$$

Therefore

$$\|\mathcal{N}(f) - \mathcal{N}(g)\|_{L^\infty_T H^{\rho-1}} \lesssim_\rho R\|f - g\|_{L^\infty_T H^\rho}$$

Since $e^{-\lambda|\xi|^2} \lesssim (1 + \lambda|\xi|^2)^{-1/2}$, it follows that

$$\left\|e^{(t-s)\Delta}\{\mathcal{N}(f) - \mathcal{N}(g)\}\right\|_{H^\rho} \lesssim_\rho R|t - s|^{-1/2}\|f - g\|_{L^\infty_T H^\rho}$$

for any $s \leq t \in [0, T]$. Thus, for any $t \in [0, T]$

$$\left\| \int_0^t e^{(t-s)\Delta}\left[\mathcal{N}(f(s)) - \mathcal{N}(g(s))\right] ds\right\|_{H^\rho} \ll RT^{1/2}\|f - g\|_{L^\infty_T H^\rho},$$

which gives the desired bounds (2) if $T \ll_\rho (1 + R)^{-2}$.

We consider the Euler equations (the Leray formulation)

$$\begin{aligned} &\partial_t u_k = \mathcal{N}_k(u), \\ &\mathcal{N}_k(u) = -\partial_a(u_a u_k) - \partial_k(R_a R_b(u_a u_b)), \end{aligned} \tag{3}$$

where $R_a = |\nabla|^{-1}\partial_a$ denote the Riesz transforms.

Theorem: (local well-posedness)
Assume $\phi \in H^\rho(\mathbb{R}^d)$, $\rho > d/2 + 1$, satisfies $\|\phi\|_{H^\rho} < R$ and the divergence-free condition $\partial_j\phi_j = 0$. Then there is $T = T(R) > 0$ and a unique solution $u \in C([0, T] : H^\rho)$ of the equation (3), which is divergence-free $\partial_j u_j(x, t) = 0$.

Moreover, the flow map $\phi \to u$ is a continuous map from the ball of radius R in $H^\rho(\mathbb{R}^d)$ to the ball of radius $2R$ in $C([0, T] : H^\rho)$.

Local well-posedness: energy estimates

The key point is the *a priori energy estimate*: assume that $u \in C([0,T] : H^\rho$ is a divergence-free solution of the Euler equation

$$\begin{aligned} &\partial_t u_k + \partial_a(u_a u_k) + \partial_k p = 0, \\ &p = R_a R_b(u_a u_b) \end{aligned} \tag{4}$$

and consider the high-order energy functional

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^d} \langle\nabla\rangle^\rho u_k(t) \langle\nabla\rangle^\rho u_k(t)\,dx,$$

where $\langle\nabla\rangle^\rho$ is given by the Fourier multiplier $\xi \to (1+|\xi|^2)^{\rho/2}$. Then

$$\begin{aligned} \partial_t E &= \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho \partial_t u_k \cdot \langle\nabla\rangle^\rho u_k\,dx \\ &= -\int_{\mathbb{R}^d} \langle\nabla\rangle^\rho (u_a \partial_a u_k) \cdot \langle\nabla\rangle^\rho u_k\,dx. \end{aligned}$$

Local well-posedness: energy estimates

Kato-Ponce inequality

$$\big\|\langle\nabla\rangle^\rho(f\partial g) - f\langle\nabla\rangle^\rho(\partial g)\|_{L^2} \\ \lesssim \|\nabla f\|_{L^\infty}\|g\|_{H^\rho} + \|\nabla g\|_{L^\infty}\|f\|_{H^\rho}. \tag{5}$$

In our case, since

$$\int_{\mathbb{R}^d} u_a\langle\nabla\rangle^\rho(\partial_a u_k)\cdot\langle\nabla\rangle^\rho u_k\,dx = 0,$$

we have

$$|\partial_t E(t)| \lesssim E(t)\|\nabla u(t)\|_{L^\infty},$$

which gives the *a priori energy estimate*

$$E(t) \le E(0) + C\int_0^t E(s)\|\nabla u(s)\|_{L^\infty}\,ds. \tag{6}$$

To prove local well-posedness we proceed in several steps:

Step 1: (parabolic regularization) We construct solutions $u^{(\nu)}$ of the regularized Navier-Stokes equation ($\nu > 0$)

$$\begin{aligned} &\partial_t u_k^{(\nu)} - \nu\Delta u_k^{(\nu)} + \partial_a(u_a^{(\nu)} u_k^{(\nu)}) + \partial_k p^{(\nu)} = 0, \\ &p^{(\nu)} = R_a R_b(u_a^{(\nu)} u_b^{(\nu)}), \end{aligned} \tag{7}$$

with the same initial data $u^{(\nu)} = \phi$. The solutions are constructed on a short time-interval $[0, T^{(\nu)}]$, with $T^{(\nu)} \approx \sqrt{\nu}$, but satisfy the same a priori energy inequality

$$E^{(\nu)}(t) \leq E(0) + C\int_0^t E^{(\nu)}(s)\|\nabla u^{(\nu)}(s)\|_{L^\infty}\,ds.$$

Since $\rho > d/2 + 1$ we have $\|\nabla u^{(\nu)}(s)\|_{L^\infty} \lesssim_\rho E^{(\nu)}(s)$.

Local well-posedness: energy estimates

Use then use Gronwall's inequality to extend the solutions $u^{(\nu)}$ to an interval $[0, T]$ where $T = T(R)$ depends only on the size of the initial data.

To summarize, we showed that for any $\nu > 0$ there is a unique solution $u^{(\nu)} \in C([0, T] : H^\rho)$ of the initial-value problem (7) that satisfies the uniform bounds

$$\|u^{(\nu)}(t)\|_{H^\rho} \leq 2R \tag{8}$$

for any $\nu > 0$ and $t \in [0, T]$.

Step 2. We would like now to let $\nu \to 0$. Look at $v := u^{\nu'} - u^{\nu}$ which satisfies the equation

$$\partial_t v_k = \mathcal{N}_k(v + u^{(\nu)}) - \mathcal{N}_k(u^{(\nu)}) + \nu' \Delta u^{(\nu')} - \nu \Delta u^{(\nu)},$$

with $v(0) = 0$. We perform energy estimates for v in L^2: define

$$\delta E(t) := \frac{1}{2} \int_{\mathbb{R}^d} v_k(t) v_k(t)\, dx.$$

Then

$$\partial_t(\delta E) = \int_{\mathbb{R}^d} v_k \partial_t v_k\, dx. \tag{9}$$

Notice that

$$\begin{aligned} &v_k[\mathcal{N}_k(v + u^{(\nu)}) - \mathcal{N}_k(u^{(\nu)})] \\ &= -v_k \partial_k [p^{(\nu')} - p^{(\nu)}] \\ &- v_k \big[(v_a + u_a^{(\nu)}) \partial_a v_k + v_a \partial_a u_k^{(\nu)}\big]. \end{aligned}$$

Local well-posedness: energy estimates

Recalling (8) and integrating by parts we have

$$\Big| \int_{\mathbb{R}^d} v_k \partial_t v_k \, dx \Big| \lesssim_{R,\rho} \delta E(t) + (\nu + \nu')\delta E(t)^{1/2}.$$

Since $\delta E(0) = 0$ it follows from (9) that

$$\sup_{t \in [0,T]} \delta E(t) \lesssim_{R,\rho} (\nu + \nu')^2.$$

if $T = T(R, \rho)$ is sufficiently small. In particular, the limit

$$u = \lim_{\nu \to 0} u^{(\nu)}$$

exists in L^2 (and in $H^{\rho'}$ for any $\rho' < \rho$). The limit $u \in C([0, T] : H^\rho)$ is a solution of the Euler equation satisfying

$$\sup_{t \in [0,T]} \|u(t)\|_{H^\rho} \leq 2R. \tag{10}$$

This gives the existence part of the local well-posedness theorem.

Step 3: (uniqueness) Assuming u, u' are regular solutions of the Euler equation with the same initial data we can let as before $v = u' - u$ which satisfies the equation

$$\partial_t v_k = \mathcal{N}_k(v + u) - \mathcal{N}_k(u).$$

We define the L^2 energy

$$\delta E(t) := \frac{1}{2}\int_{\mathbb{R}^d} v_k(t)v_k(t)\,dx,$$

and show as before that

$$|\partial_t(\delta E)(t)| \lesssim_{R,\rho} \delta E(t).$$

Since $\delta E(0) = 0$ it follows that $\delta E(t) = 0$ for all $t \in [0, T]$, thus $u = u'$ on $[0, T]$.

Step 4: (continuous dependence) The uniqueness argument shows that the flow map is continuous from the ball of radius R in H^ρ to L^2 (or to $H^{\rho'}$ for any $\rho' < \rho$, due to the uniform bounds (10)).

To prove that the map is continuous in H^ρ we need the *Bona-Smith approximations*.

If $v = u' - u$ recall that

$$\partial_t v_k = \mathcal{N}_k(v+u) - \mathcal{N}_k(u)$$

and

$$\begin{aligned} &\mathcal{N}_k(v+u) - \mathcal{N}_k(u) \\ &= -\partial_k[p' - p] - (v_a + u_a)\partial_a v_k + v_a \partial_a u_k . \end{aligned}$$

The energy estimates for the difference argument shows that

$$\|u'(t) - u(t)\|_{H^{\rho-1}} \lesssim_{R,\rho} \|u'(0) - u(0)\|_{H^{\rho-1}} \tag{11}$$

and

$$\begin{aligned} &\|u'(t) - u(t)\|_{H^{\rho}} \\ &\lesssim_{R,\rho} \|u'(0) - u(0)\|_{H^{\rho}} + \|u'(0) - u(0)\|_{H^{\rho-1}} \|u(0)\|_{H^{\rho+1}}. \end{aligned} \tag{12}$$

for any $t \in [0, T]$.
This can be combined with Littlewood-Paley projections to prove continuity in H^{ρ}.